Chapter 3. Fuzzy Sets and Fuzzy Information Processing

"Classical" engineering approaches to characterise real-world problems are essentially qualitative and quantitative ones, based on more or less accurate mathematical modelling. In such approaches expressions as "medium temperature", "big humidity", "small pressure", "very big speed", related to the variables specific to the behaviour of a **controlled plant** (**CP**), are subjected to relatively difficult quantitative interpretations. That happens because "classical" automation handles variables / information processed with well-specified numerical values. In this context the elaboration of the control strategy and its implementation in the control equipment requires an as accurate as possible quantitative modelling of the CP. Advanced control strategies (adaptive, predictive or variable structure ones) require even the permanent reassessment of the models and of the values of the parameters characterizing these (parametric) models.

Process control based on **fuzzy set theory** (often called just **fuzzy logic**) – referred to as **fuzzy control** or **fuzzy logic control** – is more pragmatically from this point of view. The reason for that concerns the capability to take over and use a linguistic characterization of the quality of CP dynamics and to adapt this characterization as function of the concrete conditions of CP operation.

L. A. Zadeh set the basics of fuzzy set theory by a paper (Zadeh, 1965) that firstly seemed to be only mathematical entertainment. The boom in the 70's in computer science opened the first prospects for practical applications of the meanwhile built theory in the field of process control / automatic control and these first applications belong to **E. H. Mamdani** and co-workers (Mamdani, 1974, Mamdani and Assilian, 1975). The reference application of fuzzy control was dealing with the use of some "special" controllers based on fuzzy set theory, **fuzzy controllers**, for cement kiln control (Holmblad and Ostergaard, 1982). In the 80's in Japan, USA and later Europe, the so-called *fuzzy boom* took place in the field of fuzzy control applications involving several domains ranging from electrical household industry up to control of vehicles, transportation systems and robots. This is caused partly by the spectacular development of electronic technology and computer systems that enabled:

- the manufacturing of circuits with very high speed of information processing, dedicated (by construction and usage) to a certain purpose including fuzzy information processing,
- the development of computer-aided design programs, which allow the control system designer to use efficiently a large amount of information concerning the controlled plant and the control equipment.

The applications of fuzzy control reported until now point out two *important aspects* related to this control strategy:

- in some situations (for example, the control of plants functional nonlinearities subject to difficult mathematical modelling or even the control of ill-defined plants), fuzzy control can be a **viable alternative** to classical, crisp control (conventional control),

- compared to conventional control, fuzzy control can be strongly based and focused on the experience of a human operator, and a fuzzy controller can model more accurately this experience (in linguistic manner) versus a conventional controller.

The main features of fuzzy control can be organised as follows:

- fuzzy control employs the so-called **fuzzy controllers** or **fuzzy logic controllers** (**FC**s) ensuring a nonlinear input-output static map that can be influenced / modified based on designer's option,
- fuzzy control can process several variables from the controlled plant, hence it can be considered as belonging to the class of multi input-multi output (MIMO) systems with interactions, therefore the FC can be viewed as a **multi input** controller (eventually, a multi output one, too), similar to state-feedback controllers,
- FCs are controllers without dynamics, but the applications and performance of FCs and **fuzzy control systems** (**FCS**s) can be enlarged significantly by inserting dynamics (derivative and / or integral components) to fuzzy controller structures resulting in the so-called **fuzzy controllers with dynamics**,
- FCs are flexible with regard to the modification of the transfer features (by input-output static maps), thus ensuring the possibility to develop a large variety of adaptive control system structures.

The approach based on human experience is acting in case of fuzzy controllers by expressing the control requirements and elaborating the control signal in terms of the "natural" IF-THEN rules belonging to the following set of rules:

...

$$\text{IF} \quad (\text{antecedent}) \quad \text{THEN} \quad (\text{consequent}), \tag{3.1}$$

...

where the **antecedent** (**premise**) refers to the found out situation concerning the CP evolution (usually compared with the desired evolution), and the **consequent** (**conclusion**) refers to the measures which should be taken – under the form of the control signal u – in order to fulfil the desired evolution. The ensemble of these rules makes up the **rule base** of the FC.

Research results obtained in studying the behaviour of the human expert emphasize that the expert has a specific strongly nonlinear behaviour accompanied by anticipative, derivative, integral and predictive effects and by adaptation to the concrete operating conditions. Colouring the linguistic characterization of CP evolution (and, accordingly, of fuzzy mathematical characterization) based on experience and translating it to the control signal elaboration and to the analysis of CP evolution (dynamics) will be characterised by parameters that enable the modification of FC features. From this point of view fuzzy control systems can be regarded as part of the general framework of **intelligent control systems**.

Fig. 3.1 presents the block diagram of principle (considered as classical in the literature) of an FCS considered as single input system with respect to the **reference input** (w) and single output system with respect to the **controlled output** (y). The second input fed to the controlled plant / fuzzy control system is the disturbance input v.

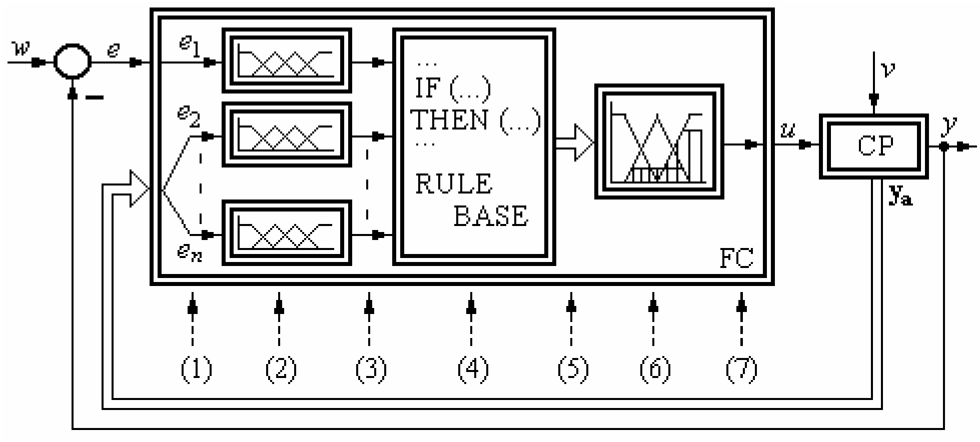

Fig. 3.1. Fuzzy control system structure.

Fig. 1.1 highlights also the operation principle of an FC in its "classical" version, characterizing ***Mamdani fuzzy controllers***, with the following **variables** and **modules**:

(1) the crisp inputs,
(2) the fuzzification module,
(3) the fuzzified inputs,
(4) the inference module,
(5) the fuzzy conclusions,
(6) the defuzzification module,
(7) the crisp output.

The essential, already mentioned, particular feature of fuzzy control systems is in the multiple interactions regarded from the plant to the controller by auxiliary variables $\mathbf{y_a}$, gathered in the vector $\mathbf{e}'$, $\mathbf{e}' = [e \quad \mathbf{y}_a^T] = [e_1 \quad e_2 \quad ... \quad e_n]^T$, that are direct or indirect inputs to the fuzzy controller. No matter how many inputs to the FC are, the FC should possess at least one input, denoted by e_1 in Fig. 3.1, corresponding to the **control error** e:

$$e_1 = e = w - y\,. \qquad (3.2)$$

According to Fig. 3.1, the operation principle of a Mamdani fuzzy controller involves the **sequence of operations** (a), (b) and (c):

(a) The crisp input information – the measured variables, the reference input (the set point), the control error – is converted into fuzzy representation. This operation is called **fuzzification** of crisp information.

(b) The fuzzified information is processed using the **rule base**, composed of the fuzzy IF-THEN rules referred to as fuzzy control rules of type (3.1) that must be well defined in order to control the given plant. The principles to evaluate and process the rule base represent the **inference mechanism** / **engine** and the result is the "fuzzy" form of the control signal u, the **fuzzy control signal**.

(c) The fuzzy control signal must be converted into a crisp formulation, with well-specified physical nature, directly understandable and usable by the actuator in order to be capable of controlling the plant. This operation is known under the name of **defuzzification**.

The three operations described briefly here characterize the three modules in the structure of an FC (Fig. 3.1), the fuzzification module (2), the inference module (4) and the defuzzification module (6), all three being assisted by an adequate database. This chapter will be dedicated to treating basics of fuzzy set theory and presenting the operation mechanisms of these three modules.

3.1. Definitions and terminology

The mathematical theory of fuzzy sets meets a tremendous development since the 70's containing chapters that are strongly related to other chapters of mathematics (Zimmermann, 1991; Driankov, et al., 1993; Kruse, et al., 1993; Passino and Yurkovich, 1998), etc. Current applications in automatic control cope with only a (small) part of the theory that proves to be relatively easily accessible to the reader. There will be presented shortly only those elements of fuzzy set theory which are considered necessary to understand at least the partial applications of fuzzy set theory to automatic control. In this context, the proofs will be omitted to enable an even easier understanding.

The essence of the fuzzy representation of information – often, but incorrectly called fuzzy information – is based on the introduction of a measure to characterize the membership of an element to a set. The **membership function** (**m.f.**) is used with this respect.

Definition 3.1: Let X be a **basic set** (**basic domain**, **universe**, **universe of discourse**) having the elements $x \in X$. The function μ_F, defined as

$$\mu_F : X \to [0;1], \tag{3.3}$$

is called **membership function** of the **fuzzy set** (**FS**) F, by which for each element $x \in X$ is mapped a value $\mu_F(x) \in [0;1]$ that characterizes the **membership degree** of x to X. A **fuzzy set** (**FS**) F, (defined) on X, is completely defined by the following set of pairs:

$$F = \{\mu_F(x)/x \mid x \in X\} = \{(x, \mu_F(x)) \mid x \in X\}, \tag{3.4}$$

where the symbol $\mu_F(x)/x = (x, \mu_F(x))$ represents the pair of discrete values "membership degree"/"crisp value belonging to the universe" and is named **singleton**. The equation (3.4) is denoted often as the sum of all $\mu(x_i)/x_i$ pairs, but the sum symbol (representing the union of these pairs on X) does not point out summation.

The definition of the FS presented in (3.4) is used in case of X being a **set with discrete elements**. In contrast, in case of a **continuous universe** an FS can be expressed as the integral of all $\mu_F(x_i)/x_i$ pairs, but the integral symbol stands for union instead of integration (Palm, et al., 1997):

$$F = \int_X \mu_F(x)/x \, \mathrm{d}x. \tag{3.5}$$

Remark: Due to the relations (3.3)-(3.5), it is fully justified, for simplicity, to use only μ_F instead of F in order to characterize an FS. This approach is used often in the literature.

The (value of the) membership degree of the (crisp) value $x = x_i$ to the above-defined FS is depicted as $\mu_F(x_i)$. The universe X can be either a compact set or a set with (finite number of) discrete elements; it will be denoted in the sequel by other symbols as well in order to connect it to the specific variables of control systems.

It has to be pointed out that for FSs the permitted values of the membership degree are within the interval [0;1] versus the case of **classical** (**crisp**) **set theory** where the membership function (i.e., the **characteristic function**) can take only two values, 0 or 1. In this case, 1 corresponds to the membership of x to X, and 0 to the contrary.

There are situations when the features of a FS are modified as function of a second independent variable denoted, for example, by t; this variable stands for the time in case of physical systems (controlled plants). Such FSs are defined and expressed according to (3.6):

$$F = \{((x,t), \mu_F(x,t)) \mid (x,t) \in X \times T\},\ \mu_F : X \times T \to [0;1]. \qquad (3.6)$$

3.2. Membership function formulation and parameterization

The following ways are employed usually to **represent an FS by means of its membership function**:

- the parametric representation under the form of an analytical function corresponding to the m.f.;
- the direct graphical representation by means of the m.f. graphics;
- the discrete representation by singletons in the case of FSs with finite number of discrete elements.

Example 3.2: To exemplify the concept of FS, it is considered the modelling of the notion (linguistic term / linguistic value) of "pleasant temperature" (in a room). Such an FS, T_p, with its m.f. μ_{Tp}, presented with continuous line in Fig. 3.2, can be associated to this notion that usually characterizes the comfort sensation. Without fully presenting the context, it is considered that the pleasant temperature is (for example) of $\theta = 22.5\,°\mathrm{C}$ in the "ideal" case, when the membership degree of the crisp value to the **linguistic term / linguistic value** (**LT**) "pleasant temperature", T_p, is equal to 1; for $\theta = 20\,°\mathrm{C}$, the membership degree is $\mu_{Tp}(20) = 0.5$.

Fig. 3.2 illustrates also with dotted line the m.f.s μ_{Ts} and μ_{Tb} of other two LTs that are adjacent to the LT T_p; these LTs representing FSs are "small (low) temperature", T_s, and "big (high) temperature", T_b, respectively.

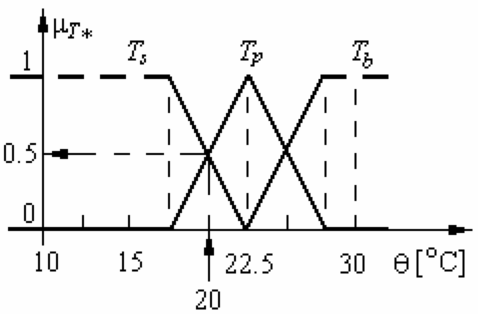

Fig. 3.2. Membership functions of linguistic terms corresponding to comfort sensation.

The example shown in Fig. 3.2 highlights the way for graphical defining the m.f.s having the continuous universe θ. Accepting that the universe has discrete values "from 0.5 °C to 0.5 °C", the notion of "pleasant temperature" can be represented on the basis of the same graphics by singletons:

$$M_{Tp} = \left\{ \frac{0}{17.5}, \frac{0.1}{18}, \frac{0.2}{18.5}, \frac{0.3}{19}, \ldots, \frac{0.9}{22}, \frac{1}{22.5}, \frac{0.9}{23}, \frac{0.8}{23.5}, \ldots, \frac{0.1}{27}, \frac{0}{27.5} \right\}. \qquad (3.7)$$

Example 3.3: The fuzzy representation of temperature θ_s in a sauna for the domain $[30\,;80]\,°\text{C}$ by means of m.f.s. In this case the temperature θ_s is the physical variable to which the **linguistic variable** (**LV**) "temperature" θ_s is assigned. This LV can describe "fuzzy" the comfort sensation in the sauna by the LTs very cold (VC), cold (C), moderate (M), warm (W), very warm (VW) and extremely warm (EW). Since the sensations "VC", "C", ..., "EW" depend on the subject who makes the assessment (depending, for example, by the degree of usage with the sauna), the same crisp value of temperature in the sauna (θ_0) can be differently assessed by different subjects or, conversely, the same LTs used by different subjects will cover and overlap different temperature domains.

Fig. 3.3 exemplifies, by means of the graphics of the membership functions corresponding to the linguistic terms VC, C, ..., EW, the comfort sensations in the sauna for three subjects: subject 1 - less accustomed to the sauna, subject 2 - relatively accustomed, but he does not like high temperatures, and subject 3 - accustomed to the sauna (e.g., he is the reference subject). It has to be outlined that the temperature $\theta_0 = 37.5\,°\text{C}$ offers to the three subjects the following sensations that may be read approximately in the graphics:

Subject 1: Warm 25%, Very Warm 75%,

Subject 2: Warm 100%,

Subject 3: Moderate 25%, Warm 75%.

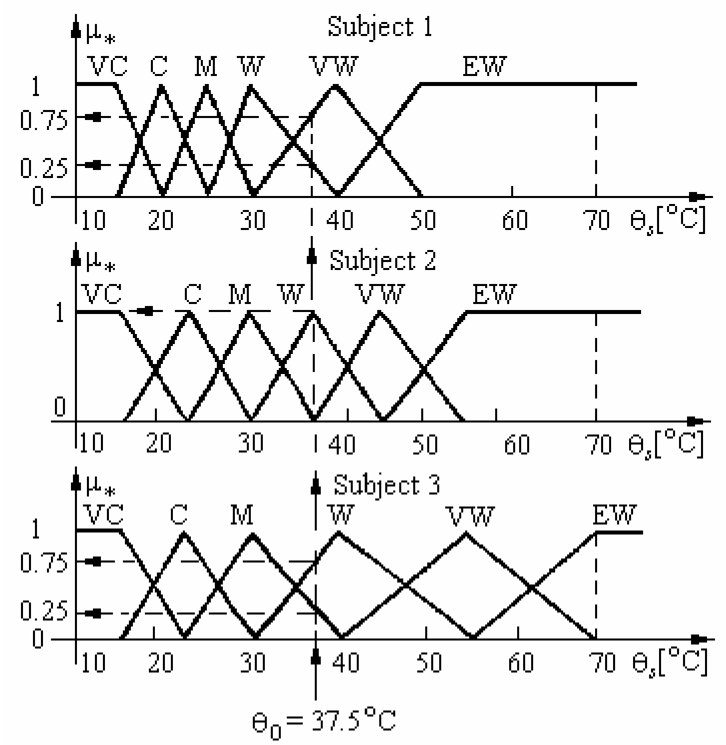

Fig. 3.3. Membership functions of linguistic terms corresponding to the linguistic variable θ_s.

It is worthwhile to be outlined also that for the same subject on the accepted universe, different LTs / comfort sensations (VC, C, ..., EW) do not have the same of m.f. shapes. Therefore, the nonlinear (and subjective) character of human perception can be emphasised.

Remark: As it can be observed in these two examples, to be shown also in the linguistic terms / linguistic values represent in fact fuzzy sets.

The most intuitive representation of m.f.s is the graphical one. Therefore, the typical m.f.s, used in the majority of fuzzy control applications, will be presented as follows. In addition, for part of the presented m.f.s relatively simple analytical relations can be associated.

The generalised bell-shaped m.f. has the graphical representation illustrated in Fig. 3.4 (a), and one of the analytical forms is

$$\mu(x) = \frac{1}{1 + \left| \dfrac{x - x_0}{a} \right|^{2b}}, \ x \in R, \ a > 0, \tag{3.8}$$

where: a – the width of the LT, x_0 – the centre, $b > 0$, with the feature $\mu(x_0) = 1$.

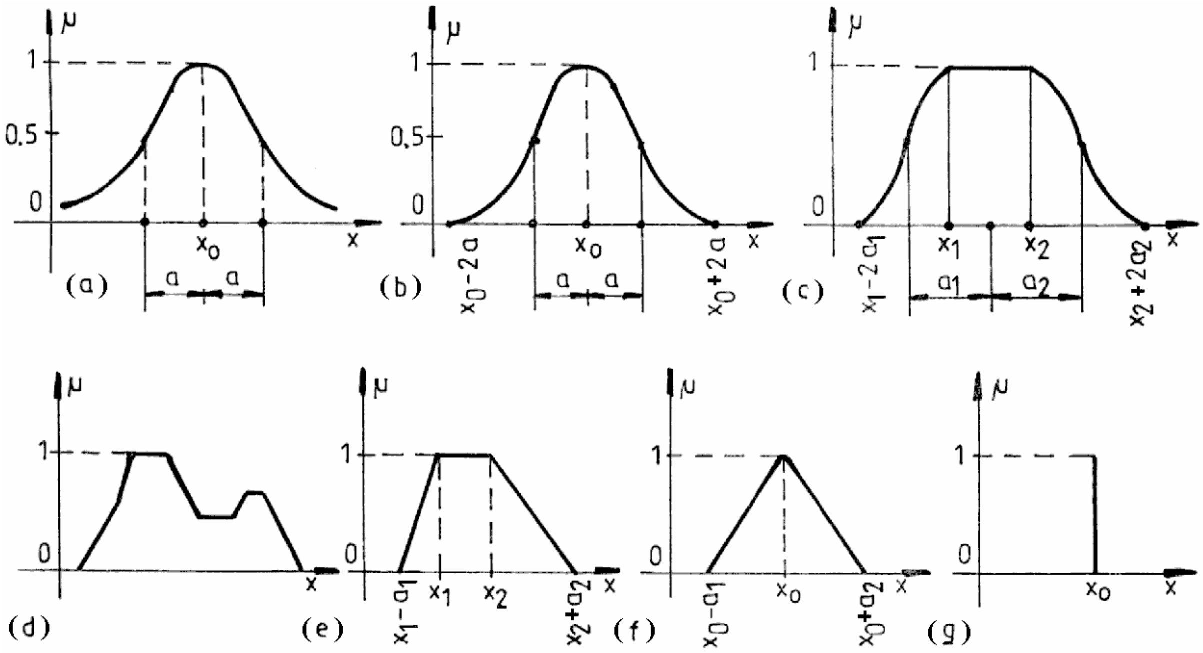

Fig. 3.4. Shapes of typical membership functions.

The Gaussian m.f. is defined in terms of the analytical expression

$$\mu_F(x) = \exp[-(x - \bar{x})^2 / 2\sigma^2], \ x \in R, \tag{3.9}$$

with the parameters $\bar{x}$ – the centre and $\sigma \neq 0$ – the width.

The cosine m.f. (generally speaking, the **trigonometric m.f.**) with symmetrical flanks, presented in Fig. 3.4 (b), has the analytical form

$$\mu(\mathrm{x})=\begin{cases} 0, & \text{if} \quad x<x_0-2a \text{ or } x<x_0+2a, \\ \frac{1}{2}\cdot\left[1+\cos\frac{\pi(x-x_0)}{2a}\right], & \text{if} \quad x_0-2a\le x\le x_0+2a, \end{cases} \tag{3.10}$$

$x\in R$, $a>0$.

M.f.s with non-symmetrical cosine flanks and enlarged core, Fig. 3.4 (c), have the analytical form (3.11):

$$\mu(\mathrm{x})=\begin{cases} 0, & \text{if} \quad x<x_1-2a_1, \\ \frac{1}{2}\cdot\left[1+\cos\frac{\pi(x-x_1)}{2a_1}\right], & \text{if} \quad x_1-2a_1\le x<x_1, \\ 1, & \text{if} \quad x_1<x<x_2, \\ \frac{1}{2}\cdot\left[1+\cos\frac{\pi(x-x_2)}{2a_2}\right], & \text{if} \quad x_2\le x\le x_2+2a_2, \\ 0, & \text{if} \quad x>x_2-2a_2, \end{cases} \tag{3.11}$$

$x\in R,\ a_1>0,\ a_2>0,\ x_1\le x_2.$

For practical fuzzy control applications there are preferred FSs / LTs which have m.f.s with line segment flanks type or singleton membership functions. These ones can be processed relatively easy analytically, and the associated m.f.s will be presented as follows.

The polynomial m.f.s with line segment flanks are presented in Fig. 3.4 (d) in a rather general form. Both convex and concave zones can be involved. Such shapes are obtained often for the fuzzy control signal (Fig. 3.1) as result of fuzzy processing at the output of the inference module.

The trapezoidal m.f. (Fig. 3.4 (e)) and its particular case, **the triangular m.f.** (Fig. 3.4 (f)), can be characterised by the following analytical representation:

$$\mu(\mathrm{x})=\begin{cases} 0, & \text{if} \quad \mathrm{x}<x_1-a_1, \\ \frac{x-x_1}{a_1}, & \text{if} \quad x_1-a_1\le \mathrm{x}\le x_1, \\ 1, & \text{if} \quad x_1<\mathrm{x}<x_2, \\ \frac{x-x_2}{a_2}, & \text{if} \quad x_2\le \mathrm{x}\le x_2+a_2, \\ 0, & \text{if} \quad \mathrm{x}>x_2+a_2, \end{cases} \tag{3.12}$$

$x\in R,\ a_1>0,\ a_2>0,\ x_1\le x_2.$

with the particular case $x_1=x_2=x_0$ corresponding to the triangular m.f.

The singleton m.f., presented in Fig. 3.4 (g), is characterised (in the context of (3.12)) by

$$x_1=x_2=x_0,\ a_1=a_2=0. \tag{3.13}$$

x_0 referred to also as **modal value**. It is important to note that a singleton can be accepted as a representation of a crisp value equal to the modal value.

The rectangular m.f. (not presented in Fig. 3.4) is used also in practice in the modules of the fuzzy controller. The analytical representation is

$$\mu(\mathrm{x})=\begin{cases}0, & \text{if} \quad \mathrm{x}<x_1, \\ 1, & \text{if} \quad x_1 \le \mathrm{x} \le x_2, \\ 0, & \text{if} \quad \mathrm{x}>x_2,\end{cases} \tag{3.14}$$

$x \in R,\ x_1 < x_2,$

where $(x_2 - x_1)$ indicates the width.

The following ***descriptors*** are associated to the analytical characterization of an FS, defined as follows for the fuzzy set $F=\{(x,\mu_F(x)) \mid x \in X\}$ (Fig. 3.5):

- **the support of an FS**, with the notation $\mathrm{Supp}(F)$ and definition

$$\mathrm{Supp}(F)=\{\mathrm{x} \mid \mathrm{x} \in X,\ \mu_F(x)>0\} \subset X, \tag{3.15}$$

- **the core of an FS**, marked by $\mathrm{K}(F)$ and defined as

$$\mathrm{K}(F)=\{\mathrm{x} \mid \mathrm{x} \in X,\ \mu_F(x)=1\} \subset X, \tag{3.16}$$

- **the height of an FS**, denoted by $\mathrm{hgt}(F)$, defined in terms of

$$\mathrm{hgt}(F)=\max\{\mathrm{x} \mid \mathrm{x} \in X\} \in [0,1]. \tag{3.17}$$

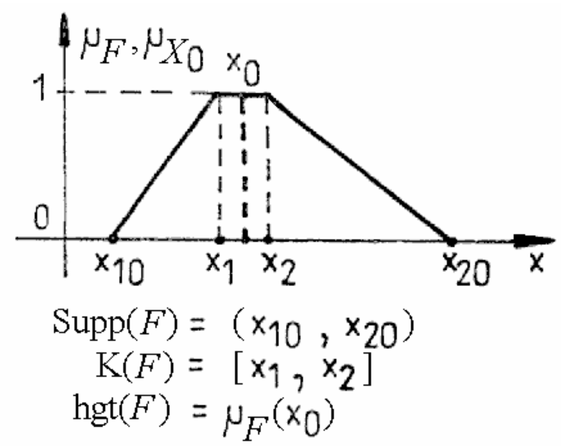

Fig. 3.5. Definitions of descriptors associated to fuzzy sets.

Remark: A **fuzzy number**, X_0, is defined in terms of (3.18) (Fig. 3.5):

$$\begin{aligned}&\mathrm{K}(X_0)=\{x \mid x \in X,\ \mu_{X0}(x)=1\}=[x_1,x_2], \\ &\mathrm{Supp}(X_0)=(x_{10},x_{20})=\{x \mid x \in X,\ \mu_{X0}(x)>0\}, \\ &\text{crisp value } x_0 \in [x_1,x_2].\end{aligned} \tag{3.18}$$

The fuzzy number represents an FS itself, and it is used in fuzzy arithmetic with applications to hardware implementation rather than to fuzzy control.

A fuzzy set $F=\{(x,\mu_F(x)) \mid x \in X\}$ is **normal** if $\mathrm{hgt}(F)=1$ and **subnormal** if $\mathrm{hgt}(F)<1$. A fuzzy set F is (**identical**) **null** is $\mu_F(\mathrm{x})=0, \forall x \in X$ and **universal** if $\mu_F(\mathrm{x})=1, \forall x \in X$.

The equality and inclusions in case of fuzzy sets are derived on the basis of crisp set theory. In this context, two fuzzy sets $A=\{(x,\mu_A(x)) \mid x \in X\}$ and $B=\{(x,\mu_B(x)) \mid x \in X\}$ are **equal**, with the nomenclature $A=B$, if each element of the universe has the same membership degree in both sets i.e.:

$\mu_A(x) = \mu_B(x),\ \forall x \in X$. (3.19)

A fuzzy set $A = \{(x, \mu_A(x)) \mid x \in X\}$ is named **fuzzy subset** of the fuzzy set $B = \{(x, \mu_B(x)) \mid x \in X\}$, with the notation $A \subseteq B$, if (Fig. 3.6)

$\mu_A(x) \le \mu_B(x),\ \forall x \in X$. (3.20)

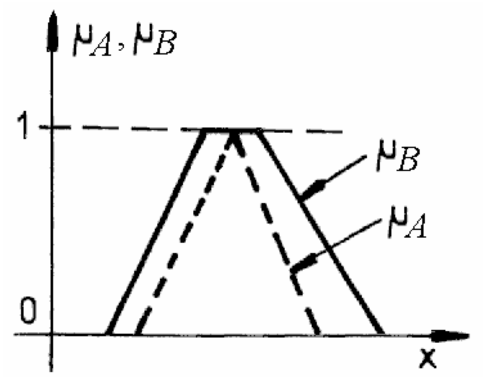

Fig. 3.6. Fuzzy set (B) and fuzzy subset (A).

A great interest in the automatic control applications is given to the feature of **fuzzy congruence** of FSs. For the sake of presenting the contents of this feature the notion of α-height cut in an FS will be presented firstly. Let $F = \{(x, \mu_F(x)) \mid x \in X\}$ be a fuzzy set and α a real number, $\alpha \in (0,1]$. The **α-height cut** in the fuzzy set F, with the notation F_α, is the following fuzzy set:

$$F_\alpha = \{(x, \mu_{F\alpha}(x)) \mid x \in X\},\ \mu_{F\alpha}(x) = \begin{cases} 1, & \text{if} \quad \mu_F(x) \ge \alpha \\ 0, & \text{otherwise} \end{cases}, x \in X, \quad (3.21)$$

and it can be viewed also as crisp set since $\mu_{F\alpha}$ can be accepted as characteristic function (Fig. 3.7).

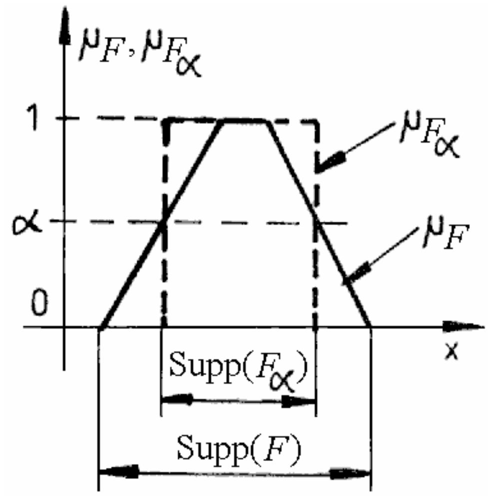

Fig. 3.7. Fuzzy set (F) and α-height cut (F_α).

The following property is valid for all fuzzy sets:

$$\mathrm{Supp}(F) \supset \mathrm{Supp}(F_\alpha)\,. \tag{3.22}$$

Also, the core of a fuzzy set equals the support of its 1-height cut ($\alpha = 1$):

$$\mathrm{K}(F) = \mathrm{Supp}(F_1) = \left\{x \in X \,\middle|\, \mu_F(x) = 1\right\}. \tag{3.23}$$

Two fuzzy sets $F_1 = \{(x, \mu_{F1}(x)) \mid x \in X\}$ and $F_2 = \{(x, \mu_{F2}(x)) \mid x \in X\}$ are **fuzzy congruent** if for any height α, $\alpha \in (0\,;\,1]$, there exist two real numbers α_1, α_2 with the feature $0 < \alpha_1, \alpha_2 \le 1$ such as

$$\mathrm{Supp}((\alpha_1 F_1)_\alpha) \subseteq \mathrm{Supp}((F_2)_\alpha),\ \mathrm{Supp}((\alpha_2 F_2)_\alpha) \subseteq \mathrm{Supp}((F_1)_\alpha)\,, \tag{3.24}$$

where the multiplication stands for the multiplication of the m.f.s. The fuzzy congruence of FSs is denoted formally by $F_1 \approx F_2$.

Concerning the fuzzy congruence it has to be outlined that two fuzzy congruent FSs, F_1 and F_2, have the same core:

$$\mathrm{K}(F_1) = \mathrm{Supp}(F_1)_1 = \mathrm{Supp}(F_2)_1 = \mathrm{K}(F_2)\,. \tag{3.25}$$

Fig. 3.8 (a) and (b) illustrate two fuzzy sets which are (a) / and are not (b) fuzzy congruent, the cores of the two FSs being different for fuzzy noncongruence (b).

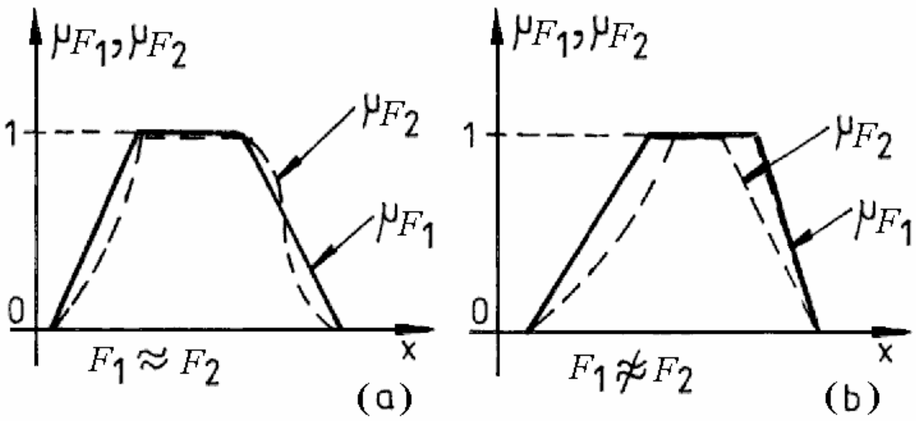

Fig. 3.8. Fuzzy congruence (a) and fuzzy noncongruence (b).

A constraint in the fuzzy congruence of two fuzzy sets is brought by conditioning the same support and the same tolerance for these FSs. This leads to the strictly fuzzy congruence. Two fuzzy sets $F_1 = \{(x, \mu_{F1}(x)) \mid x \in X\}$ and $F_2 = \{(x, \mu_{F2}(x)) \mid x \in X\}$ are **strictly fuzzy congruent** if:

$$\mathrm{Supp}(F_1) = \mathrm{Supp}(F_2) \quad \text{and} \quad \mathrm{K}(F_1) = \mathrm{K}(F_2)\,. \tag{3.26}$$

The fuzzy congruence and the strictly fuzzy congruence have some *consequences* which are remarkable from the point of view of fuzzy control:

1. The monotonous increase / decrease of the flanks of the m.f.s (for some m.f.s) represent no requirement for the strictly fuzzy congruence of FSs. Therefore in practical control applications, where the fuzzy information must be processed as quickly as possible, the flanks of m.f.s are built from straight line segments, i.e., the m.f.s will be chosen of classical shapes; triangular, rectangular, trapezoidal or singleton.

2. Major modifications in the characterization of fuzzy information are firstly obtained by modifying the support and / or core of FSs.

Another problem is related to the representation under **quantised** form of the values x (quantization of the universe / the domain of processed signal), and of the values $\mu(x)$ (quantization of the values of the m.f.). Definitely, the requirement to render back as accurate as possible a continuous signal by quantised samples is reflected by representing it using an as large as possible number of bits. Since the two mentioned quantization operations appear in the numerical representation of an FS, the following two **recommendations** are given:

- The number of levels for the representation of x and $\mu(x)$ has to be correlated; in order to overcome the modification of the support and tolerance of continuous FSs by quantised representation, the m.f. flanks must have their initial / final slope greater than 45°.
- It is recommended, similar to the case of classical quasi-continuous digital control, that the minimum number of bits for the representation of each binary word should be 12 (16).

3.3. Set-theoretic operators

In order to connect the fuzzy propositions using the logical operators AND, OR and NO, similar to the case of crisp sets, the ***set-theoretic operators*** (***operators on fuzzy sets***) intersection, union and complement, respectively are employed. The following operators are frequently used on the basis of accepting as in the previous Section the two fuzzy sets $A=\{(x,\mu_A(x))\,|\,x\in X\}$ and $B=\{(x,\mu_B(x))\,|\,x\in X\}$:

- for **intersection**, denoted by $A\cap B$, the **MIN** operator:

$$A\cap B=\{(x,\mu_{A\cap B}(x))\,|\,x\in X\},\ \mu_{A\cap B}(x)=\min(\mu_A(x),\mu_B(x)),\ \forall x\in X, \tag{3.27}$$

- for **union**, marked by $A\cup B$, the **MAX** operator:

$$A\cup B=\{(x,\mu_{A\cup B}(x))\,|\,x\in X\},\ \mu_{A\cup B}(x)=\max(\mu_A(x),\mu_B(x)),\ \forall x\in X, \tag{3.28}$$

- for **complement**, with the notation A^c, **Mamdani's fuzzy complement** operator:

$$A^c=\{(x,\mu_{Ac}(x))\,|\,x\in X\},\ \mu_{Ac}(x)=1-\mu_A(x),\ \forall x\in X\,. \tag{3.29}$$

But, as general rule, the ***triangular norms*** (***t-norms***), ***triangular conorms*** (***s-norms*** or ***t-conorms***) and ***c-norms*** (Klement, et al., 2000) are defined to represent the intersection, union and complement, respectively.

A **t-norm** is a function:

$$t:[0,1]\times[0,1]\rightarrow[0,1], \tag{3.30}$$

which fulfils the following conditions:

a) boundary (null and identity element):

$$t(0,0)=0\,,\ t(\mu,1)=t(1,\mu)=\mu\,,\ \forall\mu\in[0,1]\,, \tag{3.31}$$

b) monotonicity:

$$t(\mu_1,\mu_2)\le t(\mu_3,\mu_4)\ \text{ if }\ \mu_1\le\mu_3\ \text{ and }\ \mu_2\le\mu_4\,,\ \forall\mu_1,\mu_2,\mu_3,\mu_4\in[0,1]\,, \tag{3.32}$$

c) commutativity:

$$t(\mu_1,\mu_2)=t(\mu_2,\mu_1)\,,\ \forall\mu_1,\mu_2\in[0,1]\,, \tag{3.33}$$

d) associativity:

$t(t(\mu_1,\mu_2),\mu_3) \le t(\mu_1,t(\mu_2,\mu_3))$, $\forall \mu_1,\mu_2,\mu_3 \in [0,1]$. (3.34)

An **s-norm** is a function:

$s:[0,1]\times[0,1]\to[0,1]$, (3.35)

a) boundary (null and identity element):

$s(1,1)=1$, $s(\mu,0)=s(0,\mu)=\mu$, $\forall \mu \in [0,1]$, (3.36)

b) monotonicity:

$s(\mu_1,\mu_2) \le s(\mu_3,\mu_4)$ if $\mu_1 \le \mu_3$ and $\mu_2 \le \mu_4$, $\forall \mu_1,\mu_2,\mu_3,\mu_4 \in [0,1]$, (3.37)

c) commutativity:

$s(\mu_1,\mu_2)=s(\mu_2,\mu_1)$, $\forall \mu_1,\mu_2 \in [0,1]$, (3.38)

d) associativity:

$s(s(\mu_1,\mu_2),\mu_3) \le s(\mu_1,s(\mu_2,\mu_3))$, $\forall \mu_1,\mu_2,\mu_3 \in [0,1]$. (3.39)

The definitions of some well-accepted non-parameterised t-norms and s-norms (Zimmermann, 1991; Driankov, et al., 1993) are presented here for any $\mu_1,\mu_2 \in [0,1]$:

- the *drastic product*:

$$t_W(\mu_1,\mu_2)=\begin{cases}\min(\mu_1,\mu_2), & \text{if} \quad \max(\mu_1,\mu_2)=1,\\ 0, & \text{otherwise},\end{cases} \quad (3.40)$$

- the *drastic sum*:

$$s_W(\mu_1,\mu_2)=\begin{cases}\max(\mu_1,\mu_2), & \text{if} \quad \min(\mu_1,\mu_2)=1,\\ 0, & \text{otherwise},\end{cases} \quad (3.41)$$

- the *bounded product*:

$t_1(\mu_1,\mu_2)=\max(0, \mu_1+\mu_2-1)$, (3.42)

- the *bounded sum*:

$s_1(\mu_1,\mu_2)=\min(1, \mu_1+\mu_2)$, (3.43)

- the *Einstein product*:

$t_{1.5}(\mu_1,\mu_2)=\mu_1\mu_2/(2-\mu_1-\mu_2+\mu_1\mu_2)$, (3.44)

- the *Einstein sum*:

$s_{1.5}(\mu_1,\mu_2)=(\mu_1+\mu_2)/(1+\mu_1\mu_2)$, (3.45)

- the *Hamacher product*:

$t_{2.5}(\mu_1,\mu_2)=\mu_1\mu_2/(\mu_1+\mu_2-\mu_1\mu_2)$, (3.46)

- the *Hamacher sum*:

$s_{2.5}(\mu_1,\mu_2)=(\mu_1+\mu_2-2)/(1-\mu_1\mu_2)$, (3.47)

and it is obvious that the denominators in (3.43)-(3.47) must be nonzero.

A widely used example of t-norm is the **PROD** operator:

$\mathrm{PROD}(A,B)=\{(x,\mu_{\mathrm{PROD}(A,B)}(x)) \mid x \in X\}$, $\mu_{\mathrm{PROD}(A,B)}(x)=\mu_A(x)\mu_B(x)$, $\forall x \in X$, (3.48)

and a widely used example of s-norm is the **SUM** operator:

$$\begin{aligned}&\mathrm{SUM}(A,B)=\{(x,\mu_{\mathrm{SUM}(A,B)}(x)) \mid x \in X\},\\ &\mu_{\mathrm{SUM}(A,B)}(x)=(\mu_A(x)+\mu_B(x))/2, \ \forall x \in X,\end{aligned} \quad (3.49)$$

where the average must be taken into account in case of more fuzzy sets in order to avoid membership degrees greater than 1.

The parameterised operators fuzzy AND and fuzzy OR result in the fuzzy set $F = \{(x, \mu_F(x)) \mid x \in X\}$, with the m.f.s defined in (3.50) and (3.51), respectively:

$$\mu_F(x) = \gamma \min(\mu_A(x), \mu_B(x)) + (1-\gamma)(\mu_A(x) + \mu_B(x))/2, \ \forall x \in X, \tag{3.50}$$

$$\mu_F(x) = \gamma \max(\mu_A(x), \mu_B(x)) + (1-\gamma)(\mu_A(x) + \mu_B(x))/2, \ \forall x \in X, \tag{3.51}$$

with the parameter γ, $\gamma \in [0.1]$, ensuring the possibility of applying these operators to represent the intersection or union. Two particular values of γ transform these two operators into:

- for $\gamma = 1$: fuzzy AND becomes MIN operator to evaluate the intersection,
  fuzzy OR becomes MAX operator union to evaluate the union,
- for $\gamma = 0$: fuzzy AND and fuzzy OR become SUM operator to evaluate the union.

The operation mode of these two operators in the conditions of modifying the parameter γ is presented in Fig. 3.9.

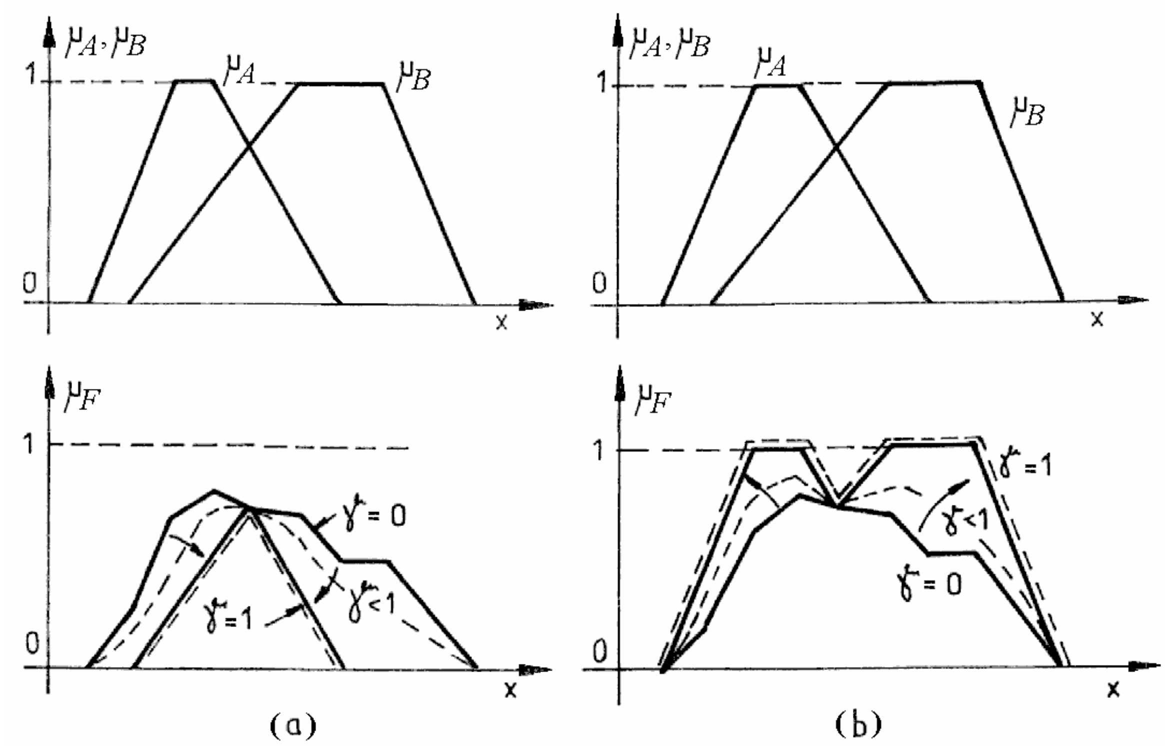

Fig. 3.9. Operation mode of fuzzy AND (a) and fuzzy OR (b) operators.

Using the same parameter γ, $\gamma \in [0.1]$, a similar possibility is ensured also in case of the **MIN-MAX** operator, which is however used seldom in fuzzy control:

$$\mu_F(x) = \gamma \min(\mu_A(x), \mu_B(x)) + (1-\gamma) \max(\mu_A(x), \mu_B(x)), \ \forall x \in X. \tag{3.52}$$

The modification of γ in the mentioned domain ensures again colouring the result. Two particular cases are:

- for $\gamma = 1$: MIN-MAX becomes MIN operator to evaluate the intersection,
- for $\gamma = 0$: MIN-MAX becomes MAX operator to evaluate the union.

Relaxing the conditions in the definitions of t-norms and s-norms leads to other categories of operators. Such operators are the uninorms, nullnorms (Fodor, 1995; Calvo, et al., 2001), the entropy- and distance-based operators (Rudas and Kaynak, 1998; Rudas and Fodor, 2006).

The ***modification operators of fuzzy sets*** represent operators based on arithmetical computations on the (m.f.s of) fuzzy sets and are mainly meant for modelling linguistic hedges as “more / less ... ”, “relatively more / less ... ”, “very …”. The particular feature of these operators is in the fact that they do not affect the support and core of the FSs, hence they ensure the strict congruence of FSs. The following frequently used modification operators are presented below, applied to the original FS $A = \{(x, \mu_A(x)) \mid x \in X\}$:

- the **concentration** operator, denoted by CON(A), with the result in a “denser” FS than the original one:

$$\mathrm{CON}(A) = \{(x, \mu_{\mathrm{CON}(A)}(x)) \mid x \in X\},\ \mu_{\mathrm{CON}(A)}(x) = (\mu_A(x))^n,\ n = 2,3,...,\ \forall x \in X, \quad (3.53)$$

- the **dilation** operator, marked by DIL(A), with the result in a “less dense” FS in comparison with the original one:

$$\mathrm{DIL}(A) = \{(x, \mu_{\mathrm{DIL}(A)}(x)) \mid x \in X\},\ \mu_{\mathrm{DIL}(A)}(x) = \sqrt[n]{\mu_A(x)},\ n = 2,3,...,\ \forall x \in X, \quad (3.54)$$

- the **contrast intensification** operator, with the notation INT(A), with the result in an FS “with intensified contrast” with respect to the original FS:

$$\mathrm{INT}(A) = \{(x, \mu_{\mathrm{INT}(A)}(x)) \mid x \in X\},$$

$$\mu_{\mathrm{INT}(A)}(x) = \begin{cases} 2(\mu_A(x))^2 & \text{if} \quad \mu_A(x) < 1/2 \\ 1 - 2(1 - (\mu_A(x))^2) & \text{otherwise} \end{cases},\ \forall x \in X. \quad (3.55)$$

3.4. Fuzzification. Linguistic variables and linguistic terms

The fundamental knowledge representation unit in fuzzy information processing is the notion of ***linguistic variable*** (**LV**), associated to the structure (N_X, T_X, D_X, M_X), where:

- N_X – symbolic name of a VL, for example N_X = speed, temperature, distance, control error, derivative / increment of control error, control signal, etc.
- T_X – set of ***linguistic terms*** / ***values*** (**LTs**), which represent the linguistic values that can take N_X. An LT associated to a VL, denoted by TL_X as element belonging to the set T_X, is a symbol for a particular property of N_X. In order to define a VL and the corresponding LTs as essential part of the fuzzification module in a fuzzy controller, Fig. 3.10 illustrates the transformation of the crisp value of furnace temperature θ_f (LV) into a fuzzy representation. The example deals with the fuzzification of an input variable. For example, in case of the temperature LV ($N_X = \theta_f$) the following linguistic terms may be defined, T_X = { VST (Very Small Temperature), ST (Small Temperature), MT (Medium Temperature), BT (Big Temperature), VBT (Very Big Temperature) }.

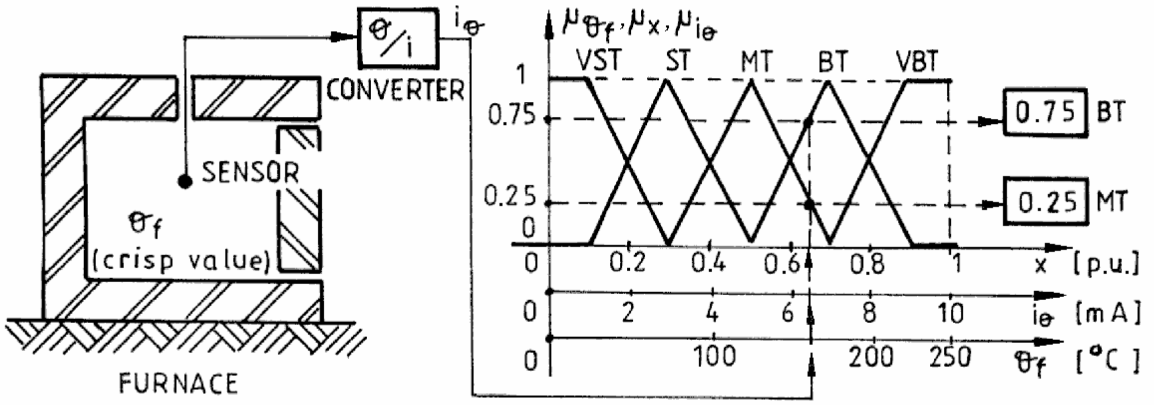

Fig. 3.10. Example to illustrate the fuzzification of an input variable.

- D_X – the domain of crisp values of the LV N_X. In case of the LV θ_f it is obtained that $D_{\theta f} = [0,250]\,^\circ\mathrm{C}$. D_X can be also the universe associated to a fuzzy set. The example presented in Fig. 3.10 highlights:
  - the necessity to define the LTs employed in the fuzzy characterization of crisp information,
  - the conversion of the variable $\theta_f \in [\,0\,;\,250\,]\,^\circ\mathrm{C}$ into its measure $i_\theta \in [\,0\,;\,10\,]\ \mathrm{mA}$; this conversion is carried out by the conventional measuring element (composed by sensor and signal converter),
  - normalizing (transforming into normalised / p.u. values) the universe by dividing to the maximum value, $\theta_{f\max} = 250\,^\circ\mathrm{C}$, in terms of

    $$x = \theta_c / \theta_{c\max}, \tag{3.56}$$

    expressed in per unit [p.u.], or normalizing the universe with respect to the nominal value of the measure of primary signal, $i_{\theta n} = 10\ \mathrm{mA}$, in terms of

    $$x = i_\theta / i_{\theta n}, \tag{3.57}$$

    for the normalised domain the result is $D_x = [0,1]$ p.u.
- M_X – semantic function which gives a meaning of an LT in terms of the elements of D_X:

$$M_X : TL_X \to \overline{TL}_X, \tag{3.58}$$

where $\overline{TL}_X$ is a fuzzy set on D_X:

$$\overline{TL}_X = \{(x, \mu_{\overline{TLX}}(x)) \mid x \in D_X\},\ \mu_{\overline{TLX}} : D_X \to [0,1]. \tag{3.59}$$

In other words, the function M_X maps a linguistic term / a symbol an interpretation expressed as fuzzy set. Proceeding this way, one may be able to make the difference from one case to another between LTs regarded as either symbols or fuzzy sets.

Remark: In order to simplify the presentation in the sequel the linguistic variables will be denoted identically to the physical variables they correspond, the meaning resulting from the context.

The crisp information concerning the evolution of controlled plant must be subject to the following transforms representing ***steps of the fuzzification module*** in order to be further processed by the inference module:

- Analog-to-digital conversion of crisp information, i.e. sampling the analog signals then quantizing the sampled signals.
- Digital processing of sampled and quantised crisp information, the processing of measured signals being accounted for here as digital filtering and eventually digital differentiation and integration.
- Processing the crisp information in a fuzzy expression in terms of LVs and corresponding LTS. Accepting that the FC inputs have well-stated values, for the fuzzy characterization of crisp information it is necessary to define the number of LTs and their m.f.s for each input LV.

The parameters in the fuzzification module to be chosen by the designer are the m.f.s of the LTs corresponding to the input LVs and eventually the scaling factors to be mentioned as follows. The literature does not provide general-purpose exhaustive recommendations with this regard, the final solution representing designer's option. Some relatively general recommendations in this context will be presented here for the sake of taking them into consideration in the development of fuzzy controllers.

1. Recommendations to choose the number of LTs corresponding to an LV. Usually this is an odd number, 3, 5 or 7. The number of LTs sets the resolution of further information processing in the FC. On the basis of several case studies, the literature proves – excepting certain special applications – that an increase in the number of LTs over 7 does not contribute significantly to an efficient increase of the resolution. However, once this number increases for each input LV, this will result in an increased number of fuzzy rules and the formulation of the rule base becomes more and more difficult. Generally the LTs are called to reflect an as general as possible contents and they depend always on the variable involved.

2. Recommendations to define and use the universe of input (linguistic) variables. It must be emphasised that the universe of input variables is pre-defined by the variation domain of sensors and interfaces (adaptation and conversion). Covering by LTs this domain will determine (in correlation with actuator's properties) the gain of the FC. The existence of several inputs determines the possibility to define around a steady-state operating point more gains, one for each input. The reasoning and corresponding mathematical characterization are the same as in case of conventional control and are similar to the definition of the *proportional band* of a conventional controller.

The universe can be defined in several ways accounting for the nature of variables involved, the most frequent ways to express it being:
- in natural units,
- in p.u., in terms of division by a (nominal or maximum) value belonging to the universe (Fig. 3.10),
- in increments with respect to a reference value, expressed in natural units or p.u.

The definition of the universe is also referred to as ***scaling***, and in case of using p.u. it is called ***input normalisation***. However, this operation must be seen in correlation with the universe of output (linguistic) variables which requires the ***output denormalisation***. Both operations are necessary in case of discrete and continuous universes as well. Besides the normalisation and denormalisation in terms of multiplying / dividing by **scaling factors** that involve nominal or maximum values belonging to the universe, other values of scaling factors can be also used in either linear or nonlinear normalisation and / or denormalisation. The scaling factors represent key parameters of the FC. So, the choice of their values is important because they affect the gains of the FCs and, further on, the dynamic performance indices of fuzzy control systems resulted after development. Furthermore, they may represent sources of instability and oscillations (Driankov, et al., 1993; Passino and Yurkovich, 1998).

To exemplify the definition of universe it is considered an example of an FC where the controlled output is a furnace temperature, θ_f, $\theta_f \in [0; 100]\,°C$. If the reference input is θ_0, then the control error $\Delta\theta$ is expressed as $\Delta\theta = \theta_f - \theta_0$ and, according to a very simple logic, $\Delta\theta$ should have the variation domain $[-100; +100]\,°C$. But, in fact, this domain will be constrained such as, in correlation with the variation domain of the control signal, to determine the controller gain. The ways for defining the universe and the number of LTs are exemplified in Fig. 3.11:

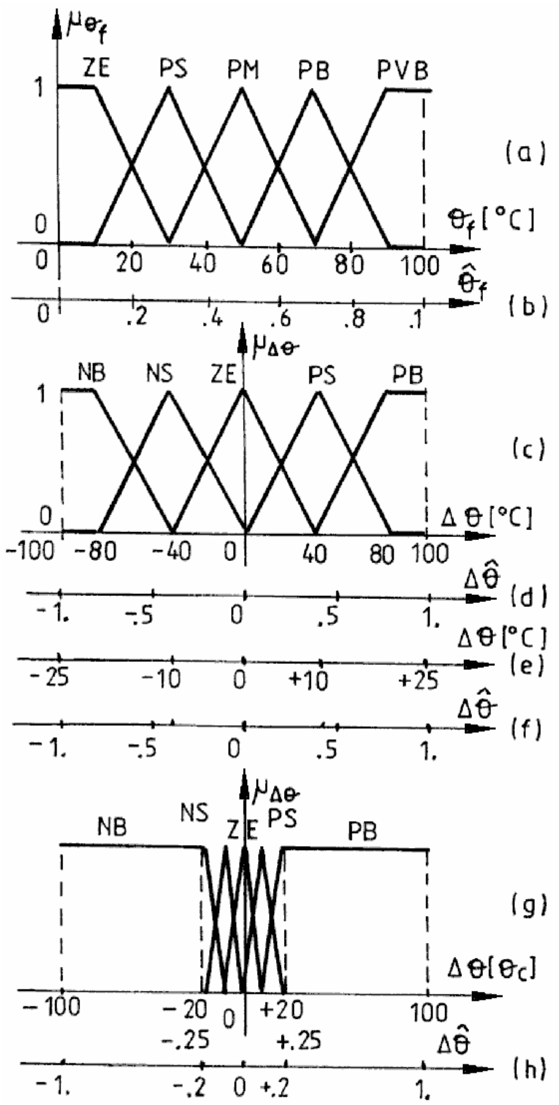

Fig. 3.11. Example to illustrate the definition of universe and number of LTs for an input LV.

(a) definition in the original signal in natural units, with the defined LTs ZE, PS, PM, PB, PVB,

(b) definition in the original signal in p.u., in terms of dividing by the maximum value $\theta_{f\max} = 100\,^{\circ}\mathrm{C}$,

(c), (d) definition in the control error $\Delta\theta$, that ensures covering the entire domain of variation, $[-100\,;+100\,]\,^{\circ}\mathrm{C}$, by LTs, and the defined LTs are ZE, PS, PB, NS, NB,

(e), (f) definition in the control error $\Delta\theta$, that ensures covering the domain $[-25\,;+25\,]\,^{\circ}\mathrm{C}$ in natural or normalised units. It can be observed in this case any exceed of the domain by the control error, $\Delta\theta < -25\,^{\circ}\mathrm{C}$ or $\Delta\theta > 25\,^{\circ}\mathrm{C}$, is treated as maximum (absolute) value of control error, of $25\,^{\circ}\mathrm{C}$. In fact, the scaling in the sense of e) corresponds to reconsidering the allocation of the m.f.s of the LTs in Fig. 3.11-c and -d according to Fig. 3.11-g and -h.

3. Recommendations for the initial choice of the m.f.s of LTs corresponding to the input LVs. In cases of no available application-oriented experience in defining the LTs and m.f.s in case of input LVs (experience gained by case studies and implementations), the following recommendations should be fulfilled in the first phase of development, Fig. 3.12 (Precup and Preitl, 1999):

- The m.f.s of the LTs corresponding to the input LVs are chosen of triangular or trapezoidal type; the m.f.s have (if possible) symmetrical shape excepting the extremity m.f.s (a), (b).
- Those allocations of m.f.s are preferred that ensure the total overlap / covering of the universe (a), (b), so each crisp value should simultaneously fire two LTs (finally, two rules). The overlap of the universe by a single LT could cause discontinuities in the input-output static map of the FC. The intersection point of the m.f.s for two adjacent (with overlap) LTs is recommended to have the ordinate greater than $0.4\,(0.45)$, excepting the extremity zones of the universe, but in case of LTs corresponding to an output LV this recommendation has to be correlated with the used defuzzification method to be presented later.
- If the LV involved is varying with ± values around zero, as it is for instance the situation of control error, symmetrical allocation with respect to zero is preferred (b);
- Zones of the universe that have no overlap by LTs are not accepted (c). No overlap creates uncertainties in the crisp control signal u, having as effect, for example, $u = 0$.
- The definition of LTs simultaneously having the core equal to one on their universe is not accepted (e).
- The too rough quantization of the m.f.s results in possible deformations of m.f. support / core (f): $\mu_{eZE} \to \mu^{\%}_{eZE}$. These deformations take place only in case of a too rough quantization of the m.f. universe and values. In this context, Fig. 3.12-f shows, for example, that a crisp value $e_0 = 0.2$ determines:
  - firing the continuous ZE LT at the value $\mu_{eZE}(e_0) = 0.98$ and the quantised ZE LT at the value $\mu^{\#}_{eZE}(e_0) = 0.98$,

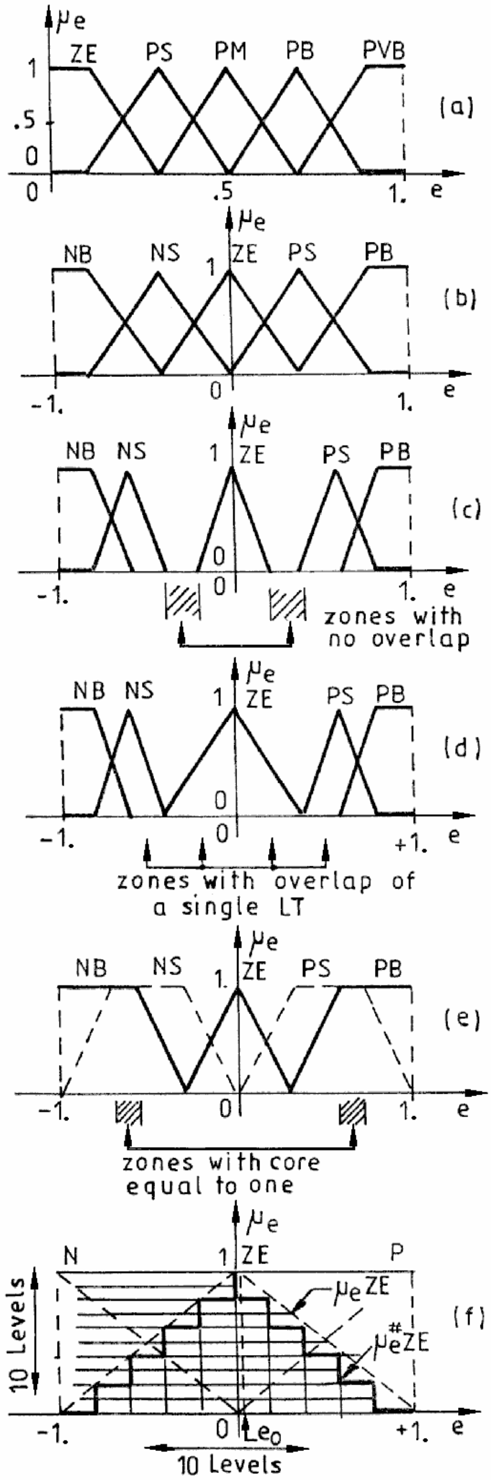

Fig. 3.12. Shapes of m.f.s and LTs corresponding to an input LV.

- firing the continuous P LT at the value $\mu_{eP}(e_0) = 0.02$ and the quantised P LT at the value $\mu^{\#}_{eP}(e_0) = 0$.

It is obvious that further information processing in the inference module will be affected by this quantization.

3.5. Inference modules. Fuzzy inference mechanisms and rule bases

The operation of an FC is based on the set of rules (3.1) representing the **rule base** that should ensure in terms of a linguistic characterization the controller operation on the universes of input and output variables. The information in the antecedent (premise) and consequent (conclusion) are expressed and then aggregated employing the operators defined in Section 3.3 and the mechanism for consequent evaluation, the **inference mechanism** / **engine**.

The dimension of rule base depends on the number of input and output LVs, on the number of LTs used in the characterization of each LV, and on the operators involved in the evaluation of intersection, union and complement in antecedent and consequent. A rule base can be:

- complete, when each crisp situation (concerning the crisp input) is covered by rules and fires at least one rule,
- incomplete, when impossible or less probably (meaningless) crisp situations for the operation of controlled plant are not defined or are left to be solved / fired by some adjacent rules,

with the observation that the completeness of a rule base will be addressed again at the end of this Section. The following relation states the **number of rules** n_R characterizing a **complete rule base**:

$$n_R = \prod_{\nu=1}^{n} n_\nu \,, \tag{3.60}$$

where n represents the number of input LVs and n_ν is the number of LTs for each input LV.

One may observe that the number of LTs of the output LV is not involved in n_R. Anyway, it is not allowed for this number to exceed n_R. For large values of n_R the correct definition of a rule base could become a problem.

Depending on the application the rule base can be formulated in terms of several expressions which are exemplified as follows.

1) The symbolic description of the general expression (3.1). This description is usually compact and clear even in the conditions of a large rule base (of course, orderly formulated). However, the formulation is space-consuming.

2) The description by the inference matrix / table (the MacVicar-Whelan diagram) or by the decision table. The table is relatively easy to formulate and interpret only in case of small numbers of input LVs, $n_I \in \{1;\, 2\,;\, 3\,;\, 4\}$. In the situation $n_I > 2$ the table becomes a table of inference tables that becomes inoperative for $n_I > 4$. This way of description has the following **outcomes**:

- it allows the easy application of the inference magnifying glass (Precup and Preitl, 1999) for the zones in which, by extending the number of LTs, it is necessary to increase the processing resolution (Fig. 3.13, where $n_{1(\mathrm{LT})}$ and $n_{2(\mathrm{LT})}$ and $n_{u(\mathrm{LT})}$ stand for the number of LTs of the variables e_1, e_2 (inputs) and u (output, usually control signal), respectively),

- it allows the easy and systematic formulation of an incomplete rule base (illustrated in the example presented in Fig. 3.14, where the empty cells correspond to situations when no rules were defined).

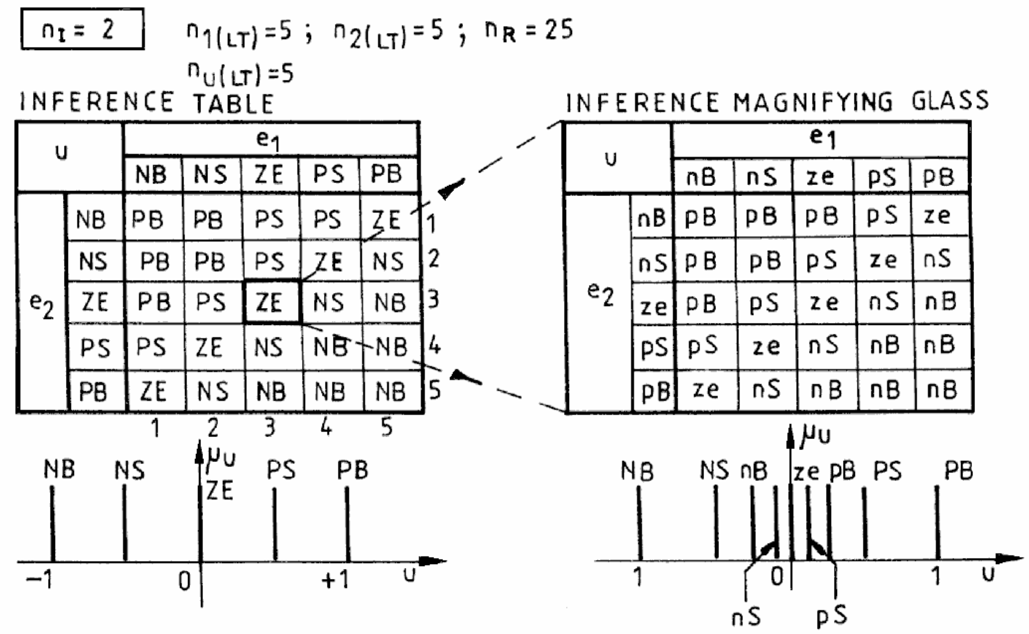

Fig. 3.13. Example of inference table for two input FC.

$n_I = 4$ $\quad n_1(LT) = 3, \ldots, n_4(LT) = 3$; $n_R = 81$

$n_u(LT) = 5$

u			e1: NB			e1: ZE			e1: PB			
		e3 →	NB	ZE	PB	NB	ZE	PB	NB	ZE	PB	
e4	NB	e2: NB		PB	PS		PB	PS		PB		1
		ZE	PB	PS	ZE	PB	ZE	NS	PS	ZE	NB	2
		PB	PS	ZE	NS	PS	NS	NB	ZE	NS		3
	ZE	NB		PB	PS	PB	PS	ZE	PS	ZE	NS	4
		ZE	PB	PS	ZE	PS	ZE	NS	ZE	NS	NB	5
		PB	PS	ZE	NS	ZE	NS	NB	NS	NB		6
	PB	NB	PB	PS	ZE		PS	ZE	PS	ZE	NS	7
		ZE	PS	ZE	NS	PS	ZE	NS	ZE	NS	NB	8
		PB		NS	NS	NS	NS	NB	NS	NB		9
			1	2	3	4	5	6	7	8	9	

Fig. 3.14. Example of inference table for four input FC.

The inference table can be formulated also in other forms. But the rules should be homogenous, i.e. they should have the same general expression (Precup and Preitl, 1999).

Once the rules are formulated, it is important to deal with the inference engine. Fuzzy relations represent the basis for the implications in fuzzy logic. Accepting the universes $X_1, X_2, ..., X_n$, the ***n-ary fuzzy relation*** is a fuzzy set R on the Cartesian product $X_1 \times X_2 \times ... \times X_n$:

$$R = \{((x_1, x_2, ..., x_n), \mu_R(x_1, x_2, ..., x_n)) \mid (x_1, x_2, ..., x_n) \in X_1 \times X_2 \times ... \times X_n\},$$
$$\mu_R : X_1 \times X_2 \times ... \times X_n \to [0,1]. \qquad (3.61)$$

Since fuzzy relations are fuzzy sets, the operators presented in the previous Section are valid for fuzzy relations, too. In addition, it is defined also the composition, denoted by $\circ$, that connects fuzzy sets and fuzzy relations. Let the fuzzy set $A = \{(x, \mu_A(x)) \mid x \in X\}$ and the fuzzy relation $R = \{((x, y), \mu_R(x, y)) \mid (x, y) \in X \times Y\}$. Then the **MAX-MIN composition** (**product**) leads to the fuzzy set B:

$$B = A \circ R = \{(y, \mu_B(y)) \mid y \in Y\}, \mu_B(y) = \max_{x \in X} \min(\mu_A(x), \mu_R(x, y)), \ \forall y \in Y. \qquad (3.62)$$

In case of fuzzy relations defined on countable or discrete universes the calculations in (3.62) corresponding to the MAX-MIN composition are done on the basis of expressing the operands in matrix forms and applying the matrix computation rules, where the classical product of elements is done using the MIN operator and the sum of products in the classical case (here it is the result of applying the MIN operator) is done using the MAX operator. Other composition versions often used in fuzzy control are:

- the **MAX-PROD composition**, where the classical product of elements is done using the PROD operator and the sum of products is done using the MAX operator,
- the **SUM-PROD composition**, where the classical product of elements is done using the PROD operator and the sum of products is done using the SUM operator.

Considering the symbolic description of the rule base where a rule is expressed in terms of

$$\text{IF } (E = A) \text{ THEN } (U = B), \qquad (3.63)$$

where E and U are input and output linguistic variables, respectively, A and B represent one LT of E and U, respectively ($A \in T_E, B \in T_U$), the rule interpretation is given by a fuzzy relation on $D_E \times D_U$, with D_E and D_U – universes of LVs E and U, respectively. The ***construction of this fuzzy relation*** proceeds according to the following steps:

1. The interpretation of the fuzzy proposition $(E = A)$, referred to as ***rule antecedent*** (***premise***), is the fuzzy set $\overline{A}$:

$$\overline{A} = \{(e, \mu_{\overline{A}}(x)) \mid e \in D_E\}, \ \mu_{\overline{A}} : D_E \to [0,1]. \qquad (3.64)$$

2. The interpretation of the fuzzy proposition $(U = B)$, called ***rule consequent*** (***conclusion***), is the fuzzy set $\overline{B}$:

$$\overline{B} = \{(u, \mu_{\overline{B}}(y)) \mid u \in D_U\}, \ \mu_{\overline{B}} : D_U \to [0,1]. \qquad (3.65)$$

3. The interpretation of the rule is given by the fuzzy relation R that ensures the ***implication***:

$$R = \{((e,u), \mu_R(e,u)) \mid (e,u) \in D_E \times D_U\},\ \mu_R : D_E \times D_U \to [0,1],$$
$$\mu_R(e,u) = \mu_A(e) * \mu_B(u),\ \forall (e,u) \in D_E \times D_U, \quad (3.66)$$

where * can be either Cartesian product or any other ***fuzzy implication*** operator. For example, in case of Mamdani's implication this operator is the MIN one.

Fuzzy propositions as those presented in (3.63) are simple ***fuzzy propositions***. However, in many fuzzy control applications the rule antecedents or consequents are composed fuzzy propositions, i.e. simple fuzzy proposition connected by means of intersection, union and / or complement operators. In this case, before step 1 and eventually before step 2 the inference mechanism should determine the m.f.s corresponding to each such proposition using adequate operators mentioned in Section 3.2.

Subsequently, the transition from fuzzy information processing using a single rule to the case of more rules, the ***Mamdani fuzzy rule bases*** consist of the rules $R^{(k)}$ according to

$$R^{(k)} : \text{IF } (E = A^{(k)}) \text{ THEN } (U = B^{(k)}),\ k = \overline{1,n}, \quad (3.67)$$

where $A^{(k)}$ is an LT corresponding to the input VL E, $A^{(k)} \in T_E$, the premise interpretation being the fuzzy set $\overline{A}^{(k)}$:

$$\overline{A}^{(k)} = \{(e, \mu_{\overline{A}(k)}(e)) \mid e \in D_E\},\ \mu_{\overline{A}(k)} : D_E \to [0,1],\ k = \overline{1,n}, \quad (3.68)$$

$B^{(k)}$ is an LT corresponding to the output VL U, $B^{(k)} \in T_U$, the conclusion interpretation being the fuzzy set $\overline{B}^{(k)}$:

$$\overline{B}^{(k)} = \{(u, \mu_{\overline{B}(k)}(u)) \mid u \in D_U\},\ \mu_{\overline{B}(k)} : D_U \to [0,1],\ k = \overline{1,n}. \quad (3.69)$$

Mamdani's implication (characterised by the usage of MIN operator in implication) is interpreted in terms of the fuzzy relation $\overline{R}^{(k)}$:

$$\overline{R}^{(k)} = \{((e,u), \mu_{\overline{R}(k)}(e,u)) \mid (e,u) \in D_E \times D_U\},\ \mu_{\overline{R}(k)} : D_E \times D_U \to [0,1],$$
$$\mu_{\overline{R}(k)}(e,u) = \min(\mu_A(e), \mu_B(u)),\ \forall (e,u) \in D_E \times D_U,\ k = \overline{1,n}, \quad (3.70)$$

where $\mu_{\overline{R}(k)}(e,u)$ stands for the ***degree of fulfilment*** (the ***firing strength***) of the rule $R^{(k)}$.

The interpretation of the rule base is done by the fuzzy relation $\overline{R}$, the result of the aggregation of all rules in terms of the union of fuzzy relations $\overline{R}^{(k)}$ afferent to the rules $R^{(k)}$:

$$\overline{R} = \bigcup_{k=1}^{n} \overline{R}^{(k)}. \quad (3.71)$$

Employing the MAX operator for the union used in ***rule aggregation*** (with the note that the "aggregation" term is used also in connecting simple fuzzy propositions as part of composed fuzzy propositions), (3.70) and (3.71) result in

$$\overline{R} = \{((e,u), \mu_{\overline{R}}(e,u)) \mid (e,u) \in D_E \times D_U\},\ \mu_{\overline{R}} : D_E \times D_U \to [0,1],$$
$$\mu_{\overline{R}}(e,u) = \max_{k=\overline{1,n}} \min(\mu_{A(k)}(e), \mu_{B(k)}(u)),\ \forall (e,u) \in D_E \times D_U. \quad (3.72)$$

In the final part of the inference module, accepting the input variable e to take the crisp value e^*, $e = e^*$, the result of applying the rule base (3.67) is interpreted as the fuzzy set $\overline{U}$ that represents the ***fuzzy conclusion*** (***control signal***):

$$\overline{U} = \{(u, \mu_{\overline{U}}(u)) \mid u \in D_U\},\ \mu_{\overline{U}} : D_U \to [0,1],$$
$$\mu_{\overline{U}}(u) = \max_{k=\overline{1,n}} \min(\mu_{A(k)}(e^*), \mu_{B(k)}(u)),\ \forall u \in D_U. \tag{3.73}$$

The interpretation of the rule base presented before corresponds to ***Mamdani's MAX-MIN composition*** (also referred to as the ***MAX-MIN inference mechanism*** or ***Mamdani's MAX-MIN compositional rule of inference***) and characterizes ***Mamdani fuzzy controllers*** / ***fuzzy inference systems***. Concluding, this inference mechanism is characterised by:

- treatment of AND linguistic connectors in the premise (the intersection of simple fuzzy propositions in the composed fuzzy proposition as part of the premise) by the MIN operator,
- treatment of OR linguistic connectors in the premise (the union of simple fuzzy propositions in the composed fuzzy proposition as part of the premise) by the MAX operator,
- implication using the MIN operator,
- rule aggregation in terms of the MAX operator.

Two other inference mechanisms are often used in fuzzy control, the MAX-PROD and SUM-PROD inference mechanisms. The ***MAX-PROD inference mechanism*** is characterised by the following operation mode:

- treatment of AND linguistic connectors in the premise (the intersection of simple fuzzy propositions in the composed fuzzy proposition as part of the premise) by the MIN operator,
- treatment of OR linguistic connectors in the premise (the union of simple fuzzy propositions in the composed fuzzy proposition as part of the premise) by the MAX operator,
- implication using the PROD operator,
- rule aggregation in terms of the MAX operator.

Finally, the ***SUM-PROD inference mechanism*** is characterised by this operation mode:

- treatment of AND linguistic connectors in the premise (the intersection of simple fuzzy propositions in the composed fuzzy proposition as part of the premise) by the PROD operator,
- treatment of OR linguistic connectors in the premise (the union of simple fuzzy propositions in the composed fuzzy proposition as part of the premise) by the SUM operator,
- implication using the PROD operator,
- rule aggregation in terms of the SUM operator.

In order to illustrate the operation mode of the MAX-MIN inference mechanism it will be presented as follows an example concerning the automatic braking of a train that approaches red lighted traffic lights, which finally implies the train stopping. The available information for decision making in braking is arranged under the form of two input variables:

- d - distance between the train and the traffic lights, constrained to $\mathsf{d} \le 1000\ \mathsf{m}$,
- v - train velocity (speed), constrained to $\mathsf{v} \le 100\ \mathsf{km/h}$.

It is assumed that the railroad and the railway engine have sensors, which can provide with sufficient accuracy the input variables. The introduction of additional conditionings could be taken into consideration in terms of two situations:

- the definition of some additional LVs (the train mass could belong, for instance, to this category),
- the suitable adaptation of the m.f.s of the LTs corresponding to the input and output LVs.

For the sake of simplicity, five LTs are assigned to each input LV, with the m.f.s defined as in Fig. 3.15, for d : VS, S, M, B, VB, for v : VS, S, M, B, VB, and the nomenclature employed S - **V**ery **S**mall, S - **S**mall, M - **M**edium, B - **B**ig and VB - **V**ery **B**ig.

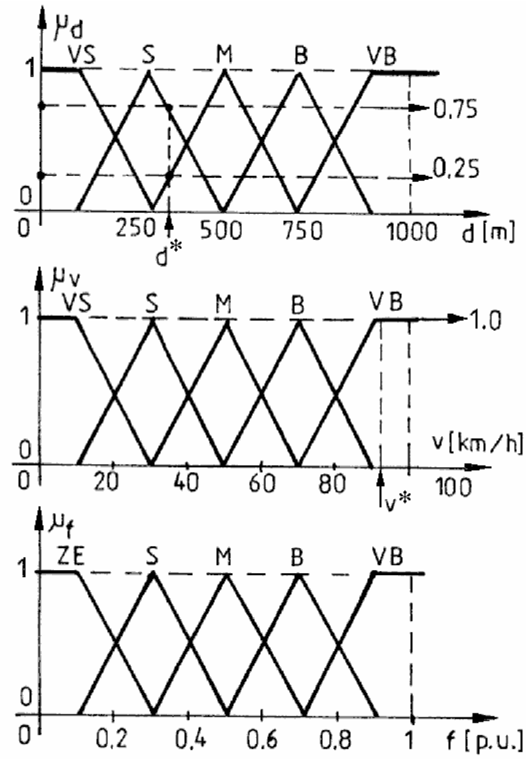

Fig. 3.15. Input and output m.f.s for train braking example.

The output considered in this example is the braking force, *f*, expressed as degree of train braking in normalised values, $f\,[p.u.]$, $f \le f_{\max} = 1$. The number of LTs defined for the output LV is five, but with the LT "ZE" (**ZE**ro), equivalent to "no braking", instead of the LT "VS".

It is accepted that he situations with *d* and *v* outside the universes are not analysed. The braking system of the train can be accounted for the choice of the defuzzification method.

The chosen shape for the m.f.s of different LTs is trapezoidal for the extremity LTs and symmetrical triangular for the middle LTs. The entire universes of distance and speed are covered and mainly overlapped by LTs, with the usual situation characterised by firing two LTs (for both *d* and *v*) and, finally, four rules.

The rule base can be defined by an expert on the basis of his / her own experience and can be more or less extended because of some extreme situations in the antecedent that may result in the same consequent. The rule base accepted here is complete (this is obvious an initial situation), consisting of 25 rules, (5 LTs corresponding to d) × (5 LTs corresponding to v), and it determines the selective firing of any of the five output LTs. The correlation antecedent $\rightarrow$ consequent for each rule and the entire rule base are illustrated in Table 3.1 as inference table / decision table.

Table 3.1. Inference table for train braking example.

f		v					
		VS	S	M	B	VB	
d	VS	M	M	B	VB	VB	1
	S	S	M	B	B	$\boxed{\text{VB}}$	2
	M	ZE	S	M	B	$\boxed{\text{B}}$	3
	B	ZE	S	S	M	B	4
	VB	ZE	ZE	ZE	S	M	5
		1	2	3	4	5	

The interpretation of data from Table 3.2.4 in order to state (word) a rule (presented as dashed) is the following:

...

$R^{(22)}$: IF $(d = S \text{ AND } v = S)$ THEN $(f = M)$ OR

...

The firing of one or another rule depends on the current (crisp) values of input variables, d^* and v^*.

Remark: If the braking process is left to the engine driver, then the input variables (e.g., at least d) become no more crisp but fuzzy and, therefore, small differences regarding the fuzzification will occur.

The operation mode of the MAX-MIN inference mechanism is exemplified here for the crisp values of current inputs $d^* = 350 \text{ m}$ and $v^* = 92 \text{ km/h}$. Fig. 3.15 demonstrates that this case leads to the following fired LTs together with the following degrees of fulfilment:

d : S-LT : $\mu_{dS}(350) = 0.75$, M-LT : $\mu_{dM}(350) = 0.25$;

v : VB-LT : $\mu_{vVB}(92) = 1$.

Therefore, only two rules, the double framed ones in Table 3.1, of the rule base will be fired:

$R^{(25)}$: IF $(d = S \text{ AND } v = VB)$ THEN $(f = VB)$ OR

$R^{(35)}$: IF $(d = M \text{ AND } v = VB)$ THEN $(f = B)$.

Remark: The value of v^* was deliberately chosen as $v^* > 90 \text{ km/h}$ in order to decrease the number of fired rules to enable a relatively easily understandable presentation of the inference engine. Nevertheless, the number of fired rules would be four in case of the chosen value $v^* \in (10; 90) \text{ km/h}$.

Finally, the fuzzy control signal is expressed under the form of a fuzzy set with the m.f. $\mu_{\bar{f}}$ according to Fig. 3.16.

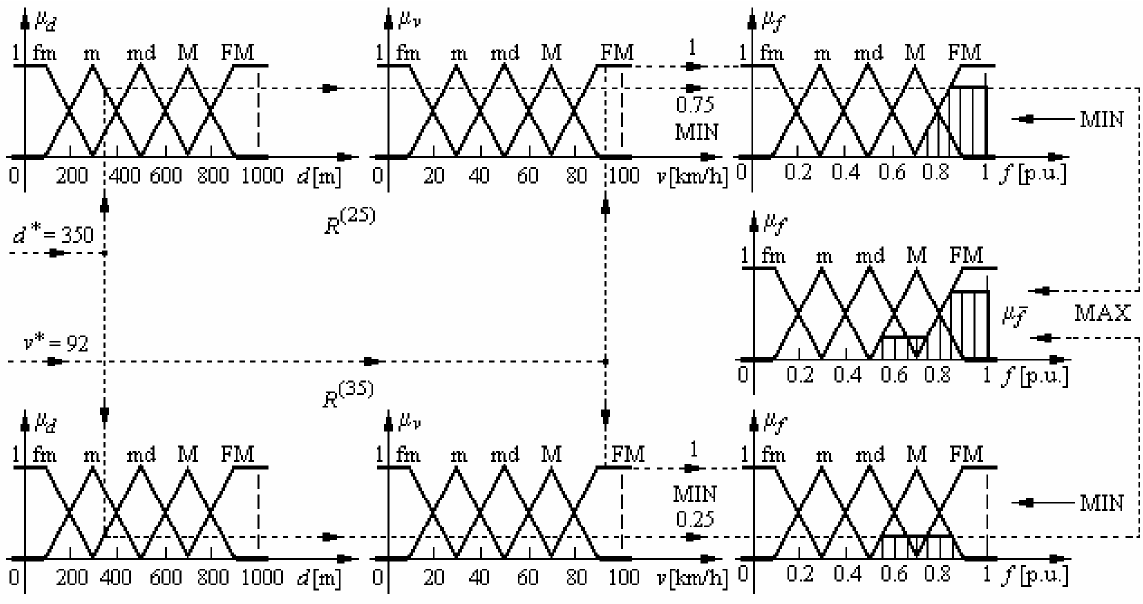

Fig. 3.16. MAX-MIN inference mechanism for train braking example.

It has to be also observed that since the braking process is related to a dynamic system (the current values of input variables are time-variable), the entire process of fuzzification and inference must always be updated by computing a new fuzzy control signal.

In case of the MAX-PROD inference mechanism the illustration of the operation mode is presented in Fig. 3.17.

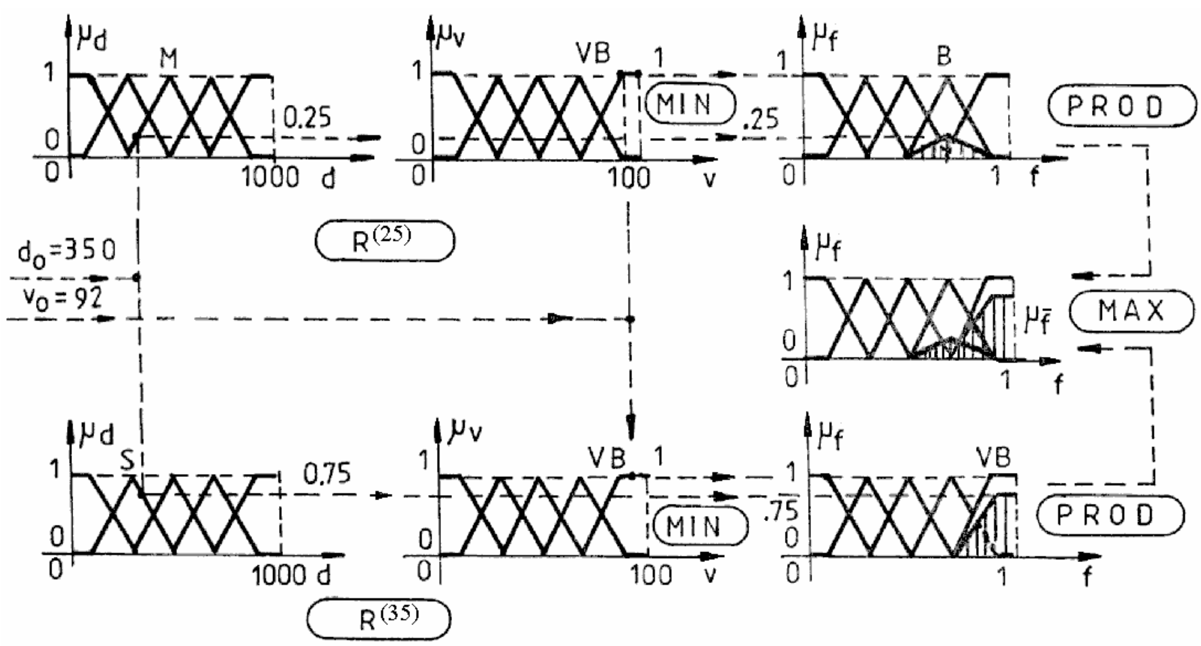

Fig. 3.17. MAX-PROD inference mechanism for train braking example.

For the sake of obtaining a better clarity the operation mode of the MAX-PROD inference mechanism / engine is also presented in Fig. 3.18-a versus the operation mode of the SUM-PROD inference mechanism shown in Fig. 3.18-b.

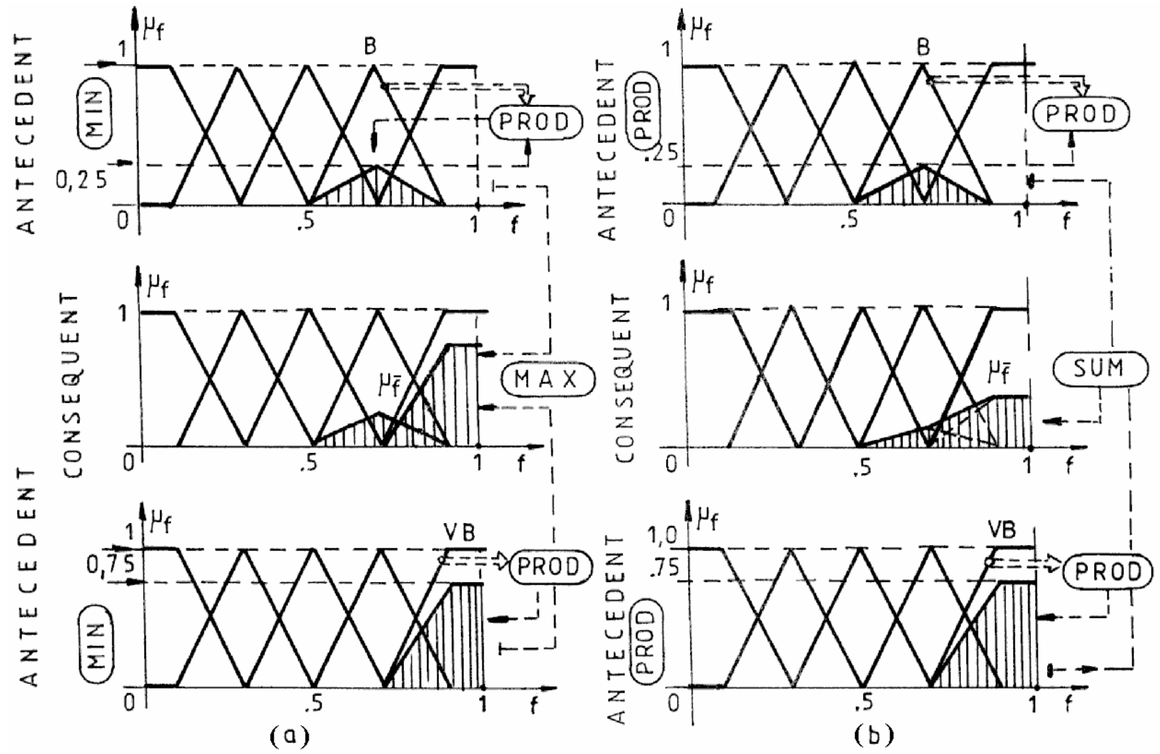

Fig. 3.18. MAX-PROD inference mechanism (a) and SUM-PROD inference meahcnism (b) for train braking example.

The inference mechanisms together with the rules presented before employ Mamdani fuzzy control rules, applicable mainly to fuzzy control with Mamdani fuzzy controllers. The ***Takagi-Sugeno fuzzy rules*** (Takagi and Sugeno, 1985) are applicable in both fuzzy control – on the basis of ***Takagi-Sugeno fuzzy controllers*** – and system modelling in case of nonlinear plants with behaviour subject to characterisation in terms of a set of operating regimes (defined in the vicinity of a set of operating points). The most general form of these rule bases is

$$R^{(k)} : \text{IF } (E = A^{(k)}) \text{ THEN } (U = f_k(E)),\ k = \overline{1,n}, \tag{3.74}$$

where the premise part is identical to Mamdani's case and the difference appears in the conclusion by the fact that $f_k(E)$ is a nonlinear or linear function that describes the dynamics of the plant / fuzzy controller or even the dynamics of the fuzzy control system for the particular value of the LT $A^{(k)}$ corresponding to the input LV E.

The **inference mechanisms** used in case of **Takagi-Sugeno** rule bases are similar to those in Mamdani's case, with the obvious difference in the rule aggregation part due to the different expressions in rule conclusions. Accepting an input variable e taking the crisp value e^*, $e = e^*$, this special form of the conclusions results in the fuzzy conclusion expressed as the (fuzzy) set of singletons (3.75):

$$\{(u_{(k)}, \mu_{(k)} = \mu_{\overline{R}(k)}(e^*, u_{(k)})) \mid (e^*, u_{(k)}) \in D_E \times D_U\},\ k = \overline{1,n}, \tag{3.75}$$

where $u_{(k)}$ is the modal value afferent to each singleton (i.e., to each fired rule) and $\mu_{(k)} = \mu_{\overline{R}(k)}(e^*, u_{(k)})$ is the firing strength of rule $R^{(k)}$.

If the LV VL E corresponds to the state vector, then $A^{(k)}$ defines a fuzzy subset of the fuzzy set on the state space corresponding to a particular operating regime and $f_k(E)$ describes the dynamics of plant / system in this operating regime. This type of fuzzy rules (associated to a certain inference mechanism) can be regarded as an interpolate mechanism that weights more or less certain local models / controllers afferent to different operating regimes depending on the current operating point.

The parameters in the inference module to be set by the designer are the rule base and the inference mechanism. The **rule base** should be defined correctly to ensure the fulfilment of the following three important **properties** that ensure a good operation of the FC:

- A **rule base** must be **complete**, i.e. any combination of input values leads to a certain output value. In relation with (3.73), this means that

$$\text{hgt}(\overline{U}) > 0, \ \forall e^* \in D_E. \tag{3.76}$$

The number of rules as part of a complete rule base (illustrated in the train braking example) equals the product of numbers of LTs corresponding to the input LVs according to (3.60).

- A rule base must be **consistent**, i.e. it does not contain contradiction. In other words, it should not contain rules with the same rule antecedent but having mutually exclusive rule consequents (the same premise leads always to the same conclusion).
- A rule base must be **continuous**, i.e., it does not contain neighbouring rules with output fuzzy sets that have empty intersection, with the note that two rules are neighbouring if the intersection of the fuzzy sets obtained by their premises (of type (3.69)) are fuzzy sets with nonzero heights (Driankov, 2001).

The **definition of rule base** becomes more and more difficult the number of inputs increases. General recommendations to define the rule base cannot be given, the role of expert (who knows the evolution of the plant) being of exquisite importance. The following **methods** are recommended by the literature as suitable for the definition or rule bases:

- the use of an expert's knowledge in controlling the given plant, with attention paid to preparing the interview,
- heuristic methods, based on engineering-like analyses of the possible evolutions of the plant variables, done in co-operation with the process engineer,
- the use of previous experimental results, real-world or simulated ones, in situation when a more or less detailed mathematical model of controlled plant is available,
- special methods employing knowledge gained from conventional controllers and identification techniques that involve clustering, (neural) network-based learning, genetic algorithms.

Concerning the **choice of inference mechanism**, the usual choice deals with one of the three mechanisms discussed here, MAX-MIN, MAX-PROD and SUM-PROD. Case studies investigated by several authors show that the inference mechanism has less significant effect on the shape of FC input-output map. Therefore, it is preferred to work with the methods that ensure good efficiency in information processing.

3.6. Defuzzification

The defuzzification is the conversion of the fuzzy control signal, which is a fuzzy set as result of the inference module, into a crisp value. The defuzzification is necessary because the actuators need crisp signals that can be interpreted in contrast with fuzzy sets.

It is obvious that the crisp value calculated by defuzzification should belong to the universe of the fuzzy control signal. In case of using a normalised domain for the control signal / output of the FC (section 3.3), the ***defuzzification module*** consists of two modules:

- the strictly speaking defuzzificatin module explained above, and
- the ***output denormalisation*** which maps the crisp value of output onto its physical domain.

A lot of defuzzification methods are used in case of ***Mamdani fuzzy controllers***. The two most often used defuzzification methods for these fuzzy controllers are the mean of maxima method and the centre of gravity method. The centre of gravity method is described in literature in several simple or complex versions.

The ***mean of maxima*** (**MoM**) ***method*** calculates the average of crisp values of output (control signal) that correspond to the conclusions with maximum firing strength. Accepting that the result of inference module is obtained under the form of the fuzzy set $\overline{u}$ on the universe D_u, that represents the fuzzy control signal, the crisp control signal u^* is obtained in terms of (3.77) and exemplified in Fig. 3.19-a:

$$u^* = 0.5[\inf_{u \in D_u}\{u \in D_u \mid \mu_{\overline{u}}(u) = \mathrm{hgt}(\overline{u})\} + \sup_{u \in D_u}\{u \in D_u \mid \mu_{\overline{u}}(u) = \mathrm{hgt}(\overline{u})\}]. \qquad (3.77)$$

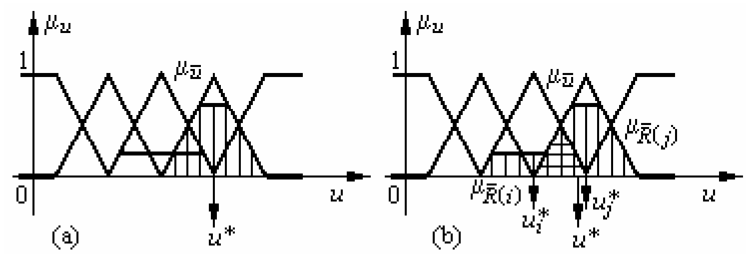

Fig. 3.19. Operation mode of mean of maxima method (a) and centre of gravity method (b).

In cases of more rules / output LTs are fired this method has a specific particular feature characterised be means of two possible situations:

- for different firing strengths corresponding to different LTs, the output LT with the maximum firing strength will be fired,
- for equal firing strengths corresponding to different LTs, an average is taken for the crisp values u_i^* corresponding to those output LTs which are fired in the inference process, for example *r* LTs, $i = \overline{1, r}$, with the result expressed in (3.78):

$$u^* = \frac{1}{p}\sum_{i=1}^{r} u_i^* . \qquad (3.78)$$

The main drawback of this method appears again when the universe of control signal is a compact set from which only some certain discrete values are fired. This may result in nonweighting the firing strengths of different rules (conclusions).

The ***centre of gravity*** (**CoG**) ***method*** determines the crisp value of output taking into consideration, in a weighted manner, all influences obtained from the rules fired by the

particular state of the inputs at a certain moment. The formulae giving the crisp control signal are adapted from mechanics and they are specific to calculating the abscissa of centre of gravity:
- in the continuous case:

$$u^* = \left[\int_{D_u} u \cdot \mu_{\bar{u}}(u)\mathrm{d}u\right] / \left[\int_{D_u} \mu_{\bar{u}}(u)\mathrm{d}u\right], \tag{3.79}$$

- in the discrete case:

$$u^* = \left[\sum_{i=1}^{m} u_i \cdot \mu_{\bar{u}}(u_i)\right] / \left[\sum_{i=1}^{m} \mu_{\bar{u}}(u_i)\right],\ m = \mathrm{card}(D_u)\,. \tag{3.80}$$

The operation mode of the CoG method is presented in Fig. 3.19-b. This illustration shows that the overlapping area is not reflected in the above calculations.

The operations in (3.79) and (3.80) are computationally rather complex and therefore result in quite slow information processing. Therefore, if the output m.f.s have favourable shapes (set by the designer), for an easy evaluation of the partial centres of gravity corresponding to different m.f.s (the symmetry of m.f.s is one of the necessary conditions imposed in this case), then the computations can become much easier. For example, in case of the symmetrical m.f.s (Fig. 3.19), from the firing strengths of several conclusions and the (fixed) centres of gravity of the resulting m.f.s, u_i and u_j, the centre of gravity of the result m.f., $\mu_{\bar{u}}$, is obtained as:

$$u^* = (\mu_{\bar{R}(i)}\, u_i^* + \mu_{\bar{R}(j)} u_j^*) / (\mu_{\bar{R}(i)} + \mu_{\bar{R}(j)})\,, \tag{3.81}$$

where $\mu_{\bar{R}(i)}$ and $\mu_{\bar{R}(j)}$ stand for the firing strengths of the conclusions of the two rules, $R^{(i)}$ and $R^{(j)}$, respectively. It is obvious that the number of fired rules can be larger than two.

The remarkable **advantages** of CoG method can be grouped as follows:
- all fired rules take part to the calculation of crisp control signal u^*, accordingly weighted with the firing strengths $\mu_{\bar{R}(i)}$ and $\mu_{\bar{R}(j)}$,
- the control signal universe can be chosen as compact domain in order to avoid the (pseudo) ripple effect caused by the discrete (crisp) values of control signal.

The CoG method presents, of course, some **drawbacks** as:
- in its primary version, already mentioned, the computation of the centre of gravity can be computationally expensive or - from one case to another - it may require the use of a relatively expensive hardware;
- the existence of some zones of the control signal universe that are not covered by m.f.s leads to peculiar shapes of the input-output static maps of fuzzy controllers;
- the existence of the ***extremity problem*** that occurs when for an inadequate choice of the shapes of m.f.s corresponding to the extremity output LTs (with respect to the control signal universe), this problem is characterised by the fact that the minimum value $\hat{u}_{min} = 0$ or the maximum one $\hat{u}_{max} = 1$ cannot be reached. In such situations the CoG method leads to the minimum value $\hat{u}_{min}$ and the maximum one $\hat{u}_{max}$ taking possible values $\hat{u}_{min} > 0$ and $\hat{u}_{max} < 1$, respectively. One situation is illustrated in Fig. 3.20 (for the accepted case of working with normalised values) and will be addressed again in this section together with solutions to avoid it.

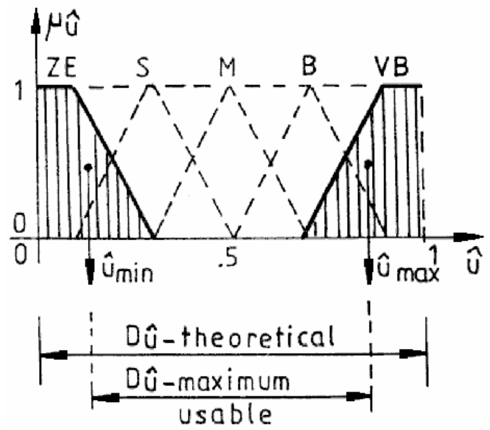

Fig. 3.20. Extremity problem in case of CoG method ($D_{\hat{u}}$ – domain of crisp values resulted for the control signal $\hat{u}$).

The application of CoG method becomes much simpler for convenient choices of the shapes of m.f.s corresponding to the output LTs. The computations in such versions with singleton and rectangular m.f.s are presented in (Precup and Preitl, 1999).

Another defuzzification method, proposed as generalisation of MoM and CoG methods, is the ***BADD method*** (Filev and Yager, 1991). The crisp control signal u^* in case of this method is calculated as follows:

- in the continuous case:

$$u^* = \left[\int_{D_u} u(\mu_{\bar{u}}(u))^{\delta} \, \mathrm{d}u \right] / \left[\int_{D_u} (\mu_{\bar{u}}(u))^{\delta} \, \mathrm{d}u \right], \qquad (3.82)$$

- in the discrete case:

$$u^* = \left[\sum_{i=1}^{m} u_i (\mu_{\bar{u}}(u_i))^{\delta} \right] / \left[\sum_{i=1}^{m} (\mu_{\bar{u}}(u_i))^{\delta} \right], \ m = \mathrm{card}(D_u), \qquad (3.83)$$

where the parameter δ sets the defuzzification method involved: $\delta = 1$ for CoG and $\delta = 0$ for MoM.

The most used defuzzification method in case of ***Takagi-Sugeno fuzzy controllers / systems*** is the weighted area method. Accepting the fuzzy control signal in (3.75), use is made of the following relationship to calculate the crisp control:

$$u^* = \left[\sum_{k=1}^{n} u_{(k)} \cdot \mu_{(k)} \right] / \left[\sum_{k=1}^{n} \mu_{(k)} \right]. \qquad (3.84)$$

The parameters in the defuzzification module to be set by the designer in case of Mamdani FCs are the m.f.s of the LTs corresponding to the output LV, the defuzzification method and the conversion of crisp signal. Analyses must be done that enable setting all these parameter values. Any analysis should account for:

- the characteristics of the actuators, which are without dynamics or with negligible dynamics, eventually absorbed by the controlled plant,
- the minimum control system performance indices to be achieved by the fuzzy control system,
- the version of FC implementation, through hardware or software construction.

Aspects concerning the development of any FC from the point of view of setting the parameters in the defuzzification module will be presented as follows.

1. Aspects concerning the choice of linguistic terms and membership functions corresponding to the output / control signal linguistic variable. The main aspects of interest in this case can be resumed as follows:

a) The number of chosen LTs is usually odd (3, 5, or possibly 7). A larger number of LTs brings no spectacular results in the shape of the input-output static map. It is usual practice, for the sake of ensuring a good sensitivity of the FC, to apply in some certain operating points (e.g. in the vicinity of the zero value for the control error) the principle of inference magnifying glass according to section 3.5.

b) The existence of zones in the universe with no covering by LTs / m.f.s does not represent a serious problem. The covering of universe by continuous or discrete crisp values of the control signal is solved in terms of the convenient choice of defuzzification method.

c) The scaling / definition of the universe. The universe has to be always scaled / defined in such a way that it should fulfil the control requirements of actuator (A), and the output denormalisation mentioned in section 3.4 and in this section should be also taken into consideration. This means that:

- the crisp control signal is not permitted to exceed the extreme values (u_m , u_M) accepted by the actuator, this exceed is related to: turning off / on the A, dynamic forcing to obtain an as reduced as possible settling time, etc.,
- the variation domain of signal fed to the actuator, $D_{u\,(A)}$, and the variation domain of actuating signal, D_m, must be overlapped sufficiently by the variation domain of control signal, D_u .

This last remark can also become a specific aspect known as **the extremity problem** of the FC. As already mentioned, the incorrect definition of the m.f.s of output LTs in correlation with an inadequate chosen defuzzification method can result in the situation when there is no overlap of the necessary domain $D_{u\,(A)}$ by the obtained crisp control signal.

The extremity problem is extremely important since it is necessary to guarantee that the fuzzy control system will reach the imposed operating points as necessary condition in achieving the desired / imposed control system performance indices and will be exemplified here by a stabilised electro hydraulic actuator (A) consisting of an electro hydraulic converter (EHC) and a servomotor (SM) controlled by an FC (Fig. 3.21-a). Five LTs are considered for the LV "fuzzy control signal": ZE, PS, PM, PB, PVB. The defuzzification at the level of the FC is ensured by the CoG method.

The essential condition imposed to actuator is:

- for $u_0 = u_{min}$ (0) $\rightarrow$ A turned off,
- for $u_0 = u_{max}$ (1) $\rightarrow$ A turned on.

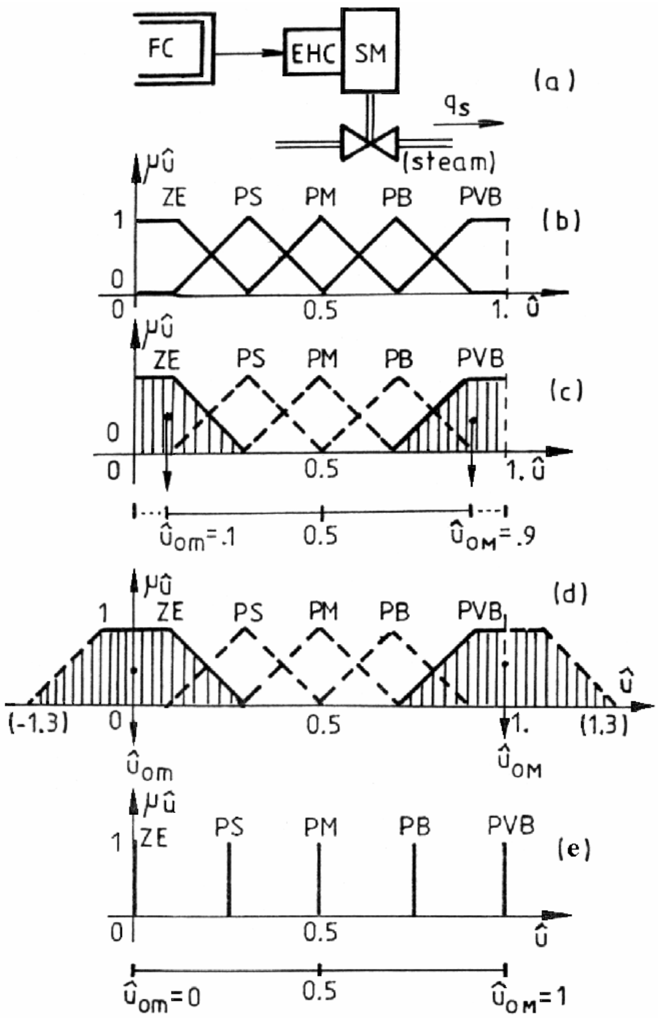

Fig. 3.21. Extremity problem and solutions to avoid it.

In order to fulfil this requirement it is necessary to define correctly the m.f.s of the LTs corresponding to the output LV. The classical definition presented till now and shown in Fig. 3.21-b seems to be acceptable, but it proves to be unsuitable because when firing only the extremity LTs (ZE, PVB), the following results are obtained: $\hat{u}_{0m} = 0.1$ and $\hat{u}_{0M} = 0.9$ (Fig. 3.21-c).

To **avoid** this kind of situations several measures can be taken, only the following ones being viable:

- the symmetrical extension the extremity m.f.s, Fig. 3.21-d,
- the modification of the shapes of m.f.s, e.g., to singleton m.f.s, Fig. 3.21-e,
- the choice of another defuzzification method, overall or just in the extremity zones.

The order of application of these measures has to be analysed depending on all other factors concerning the FC.

d) The shapes of m.f.s will be chosen to ensure - in correlation with the defuzzification method - maximum efficiency in information processing (usually reflected in an as reduced as possible computation time). As it has already been mentioned, the following forms as part of usual m.f. forms (presented in section 3.3) are recommended to be used for the LTs of output LVs:

- singleton m.f.s,
- rectangular m.f.s,
- triangular and trapezoidal m.f.s.

The following remarks are important in relation with these recommended m.f.s:

- the singleton m.f.s are the easiest ones to be processed,
- the rectangular m.f.s modify significantly the amount of computations, but, by varying the width of the rectangle (the support), additional modifications of FC features are obtained, nevertheless the overlap problem can be avoided in comparison with the situation of triangular m.f.s (the computation of the centre of gravity becomes heavier),
- on the basis of all aspects presented before, the triangular m.f.s seem to be the most unfavourable ones.

2. Aspects concerning the choice of defuzzification method. The following criteria have to be accounted for when choosing the defuzzification method (Driankov, et al., 1993; Precup and Preitl, 1999):

- The choice of the defuzzification method depends on the type of actuator:
  - in case of an actuator A with finite number of discrete states the MoM method is recommended,
  - in case of an actuator with compact variation domain / universe the CoG method is preferred.
- The continuity of the input-output static of FC has to be ensured. This means that a small change in the fuzzy control signal should not result in a large change in the crisp control signal. The CoG method satisfies this criterion and the MoM method does not.
- The disambiguity must be offered. This is characterised by avoiding situations when two relatively large areas in the m.f.s of fuzzy control signal are covered by two areas in the m.f.s of fuzzy sets as result of implication. Both CoG and MoM methods satisfy this criterion.
- The plausibility is necessary. This is characterised by placing the crisp control signal approximately in the middle of the support of fuzzy control signal. The CoG does not satisfy this criterion and the MoM satisfies this criterion only in the case of MAX-PROD inference mechanism.
- The computational complexity is particularly important in practical applications of fuzzy controllers. The MoM is a computationally fast method, whereas the CoG method is much slower. Although the use of CoG seems to be difficult as it has already been mentioned, choosing particular shapes of m.f.s and well-acknowledged inference methods determines:
  - faster information processing,
  - fuzzy control signals with m.f.s having convenient shapes that enable relatively easy analytical calculations.

3. Conversion of crisp control signal. Depending on the defuzzification method and on the type of actuator, the crisp control signal will be (in case of digital control):

- the current crisp value of control signal u_k or

- the increment of current crisp value with respect to the previous crisp value of control signal, $\Delta u_k = u_k - u_{k-1}$.

In both situations the resulted value is converted into analog form by a digital-to-analog converter, excepting the situations when the actuator accepts directly as input the binary form of the crisp control signal from the case of continuous process control. The problems and results related to information quantization are the same as in the case of analog-to-digital conversion (section 3.4).

Concluding, since the focus in this chapter was mainly on Mamdani fuzzy systems / controllers, the following section will be dedicated to a short presentation of the operation mechanisms in case of Takagi-Sugeno fuzzy models and Tsukamoto fuzzy models.

3.7. Takagi-Sugeno fuzzy models and Tsukamoto fuzzy models

The ***Takagi-Sugeno fuzzy models***, known also as ***Takagi-Sugeno-Kang*** (***TSK***) ***fuzzy model***s or ***Sugeno models*** (Takagi and Sugeno, 1985; Sugeno and Kang, 1988), have been suggested firstly as an alternative to the development of systematic approaches capable of generating fuzzy rules from a given input-output data set. Considering a two input-single output system, a typical fuzzy rule in a Takagi-Sugeno fuzzy model has the following form which is similar to the already presented rule in (3.74):

$$\text{IF } (x = A \text{ AND } y = B) \text{ THEN } (z = f(x, y)), \tag{3.85}$$

where A and B are FSs in the premise (antecedent) and $f(x, y)$ is a crisp function in the conclusion (consequent). Usually this function is a polynomial in the input variables x and y, but it can be any linear or nonlinear function as long as it can appropriately describe the output of the model, z, in the fuzzy region specified by the rule antecedent. When this function is a first-order polynomial, the resulting fuzzy model (fuzzy inference system) is referred to as ***first-order Sugeno fuzzy model***. When $f(x, y)$ is a constant (in fact, more constants, each one appearing in a certain rule), the fuzzy model is called ***zero-order Sugeno fuzzy model***, a special case of Mamdani fuzzy inference system described in this chapter. In case of zero-order Sugeno fuzzy models the rule consequent is specified by a singleton or a pre-defuzzified consequent.

As mentioned earlier in this chapter, the **inference mechanisms** involved in Takagi-Sugeno fuzzy models are similar to those in Mamdani fuzzy models (fuzzy inference systems), with the difference in the rule aggregation part due to the different expressions in rule consequents. This determines the fuzzy conclusion expressed as the (fuzzy) set of singletons of type (3.75). The most used **defuzzification method** in case of Takagi-Sugeno fuzzy models is the weighted area method where the calculation of the crisp control signal is of type (3.84). The other modules in Takagi-Sugeno fuzzy model structure are identical to Mamdani's case presented before. Due to these features the outputs of Takagi-Sugeno fuzzy models are smooth functions of their input variables as long as the neighbouring m.f.s in the antecedent have enough overlap whereas the overlap of m.f.s in the consequents of Mamdani fuzzy models does not have a decisive effect on the smoothness (the overlap of the antecedent m.f.s plays the key role with this respect).

The operation mode of a Sugeno fuzzy model is illustrated in Fig. 3.22 for a first-order Sugeno fuzzy model. Usually the t-norm used in the inference engine and highlighted in Fig. 3.22 is the MIN or PROD operator.

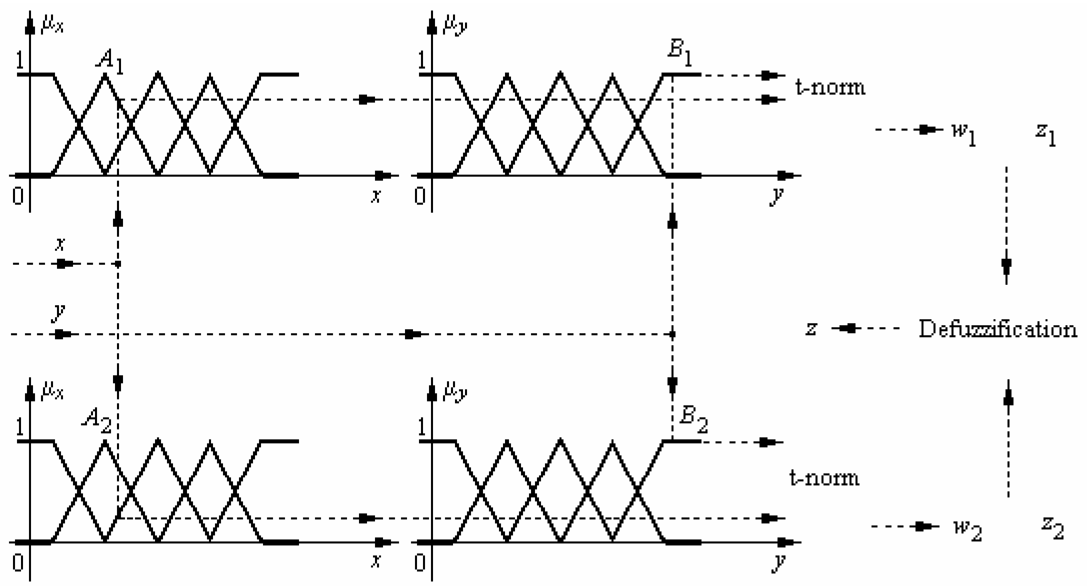

Fig. 3.22. Operation mode of two input-single output first-order Sugeno fuzzy model.

The parameters w_1 and w_2 stand for the firing strengths of the two rules, and the rule consequents in the two rules, z_1 and z_2, that represent the fuzzy consequent, are accepted to be expressed as:

$$z_1 = p_1 x + q_1 y + r_1,\ z_2 = p_2 x + q_2 y + r_2 . \tag{3.86}$$

Subsequently, the crisp output of the fuzzy model, z, is obtained in terms of the weighted area method of defuzzification:

$$z = (w_1 z_1 + w_2 z_2)/(w_1 + w_2) . \tag{3.87}$$

Since each rule has a crisp output (z_1 and z_2 here), it is often considered in the literature that Takagi-Sugeno fuzzy models do not possess defuzzification modules, the defuzzification being replaced with several operators (Jang, et al., 1997). In case of the weighted area method of defuzzification, this operator (defined in (3.87)), is known as the ***weighted average*** operator. In practice, the weighted average operator is sometimes replaced with the ***weighted sum*** operator in order to reduce the computational costs. This approach is used especially in case of fuzzy modelling and identification and is expressed in (3.88) as far as Fig. 3.22 is concerned:

$$z = w_1 z_1 + w_2 z_2 . \tag{3.88}$$

The ***Tsukamoto fuzzy models*** (Tsukamoto, 1979) are characterised by special rule consequents represented using fuzzy sets with monotonically membership functions. As a result, the inferred output of each rule is defined as a crisp value induced by the rule's firing strength. Then, the crisp output of the fuzzy model, z, is obtained in terms of the weighted area method of defuzzification or, in other words, in terms of taking the weighted average of each rule's output.

The operation mode of a Tsukamoto fuzzy model is illustrated in Fig. 3.23, where the defuzzification is applied according to (3.87), similar to the case of Takagi-Sugeno fuzzy models. The application of the weighted average method of defuzzification avoids other time-consuming defuzzification methods. However, Tsukamoto fuzzy models are not used often since it is not as transparent as Mamdani and Takagi-Sugeno fuzzy models.

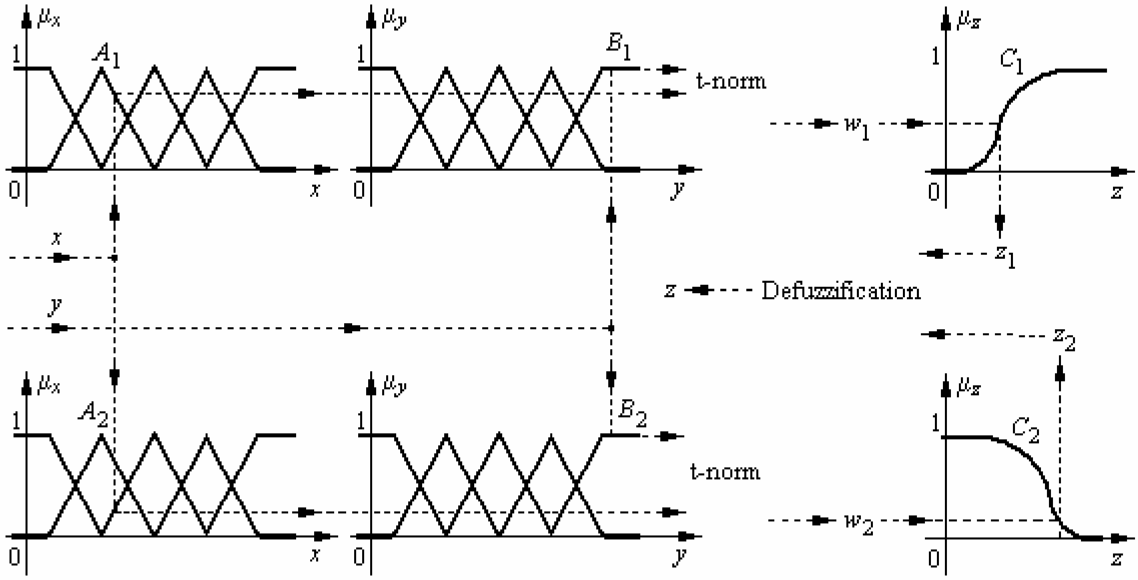

Fig. 3.23. Operation mode of two input-single output Tsukamoto fuzzy model.

Remark: Zero-order Sugeno fuzzy models represent also special cases of Tsukamoto fuzzy models where each rule consequent is specified by a step function m.f. centred at the constant involved in the rule consequent.

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